Green's Functions of the Laplacian

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1. Preliminary Calculations. Suppose we are on $\mathbb{R}^n$, with $n \geq 3$. Let

$$, (x,y) = \omega_{n-1}|x-y|^{2-n}$$

where ω_{n-1} is the volume of the (n-1)-sphere S^{n-1}. We want to compute derivatives of this function.

$$\begin{aligned}
\left|\frac{\partial}{\partial x^i}, (x,y)\right| &= \left|\omega_{n-1}\frac{\partial}{\partial x^i}\left(\sum_j (x^j-y^j)^2\right)^{(2-n)/2}\right| \\
&= \omega_{n-1}(n-2)\left|\frac{x^i-y^i}{\left(\sum_j (x^j-y^j)^2\right)^{n/2}}\right| \\
&\leq \omega_{n-1}(n-2)\frac{r}{r^n} \\
&= \omega_{n-1}(n-2)\frac{1}{r^{n-1}}
\end{aligned}$$

and $, (x,y) = \omega_{n-1}r^{2-n} \leq \omega_{n-1}r^{1-n}$ for $r \leq 1$ so for a bounded domain Ω we have that $, (x,y)$ and $\frac{\partial}{\partial x^i}, (x,y)$ are bounded by $(n-2)r^{1-n}$, which is integrable over Ω, so we can interchange the differentiation and integration, so $\frac{\partial}{\partial x^i}\int_\Omega , (x,y)f(y)dy = \int_\Omega \frac{\partial}{\partial x^i}, (x,y)f(y)dy$.

Now,

$$\begin{aligned}
\frac{\partial^2}{\partial x^k \partial x^i}, (x,y) &= \omega_{n-1}\frac{\partial}{\partial x^k}\left[(n-2)\frac{x^i-y^i}{\left(\sum_j (x^j-y^j)^2\right)^{n/2}}\right] \\
&= \left\{\begin{array}{ll}
\omega_{n-1}(2-n)n\frac{(x^i-y^i)(x^k-y^k)}{\left(\sum_j (x^j-y^j)^2\right)^{(n+2)/2}} & \text{if } i \neq k \\
\omega_{n-1}(2-n)n\frac{(x^i-y^i)^2}{\left(\sum_j (x^j-y^j)^2\right)^{(n+2)/2}} + (n-2)\frac{1}{\left(\sum_j (x^j-y^j)^2\right)^{n/2}} & \text{if } i = k
\end{array}\right\}
\end{aligned}$$

so $\left|\frac{\partial^2}{\partial x^k \partial x^i}, (x,y)\right| \leq 2\omega_{n-1}n(n-2)\frac{1}{r^n}$, but this is not integrable, so we cannot simply interchange the order of integration to get $\Delta \int , (x,y)f(y)dy$. We need to cut off the singularity.

2. Two proofs of $\Delta_{\text{distr}(y)}, (x,y) = \delta_x(y)$ on $\mathbb{R}^n$. Let's first do it directly. Consider $\int , (x,y)\Delta f(y)dy$. We want to use the divergence theorem (but can't for $x = y$), so let's look at $\int_{\mathbb{R}^n\setminus B_\epsilon} \text{div}_y \left[, (x,y)\nabla f(y)\right] dy = \int_{\partial B_\epsilon} , (x,y)\nabla f(y)\cdot \nu(y)ds(y)$ where ν is the outward pointing normal, and $B_\epsilon = B_\epsilon(x)$. Notice that the left hand side is

$$\int_{\mathbb{R}^n\setminus B_\epsilon} \left[\nabla_y , (x,y)\cdot \nabla f(y) + , (x,y)\Delta f(y)\right] dy$$

If we also look at

$$\int_{\mathbb{R}^n\setminus B_\epsilon} \text{div}_y \left[\nabla_y , (x,y) f(y)\right] dy = \int_{\partial B_\epsilon} f(y)\nabla_y , (x,y)\cdot \nu(y)ds(y)$$

we can subtract the two and get

$$\int_{\mathbb{R}^n\setminus B_\epsilon} \left[, (x,y)\Delta f(y) - \Delta_y , (x,y) f(y)\right] dy = \int_{\partial B_\epsilon} \left[, (x,y)\nabla f(y)\cdot\nu(y) - f(y)\nabla_y , (x,y)\cdot\nu(y)\right] ds(y)$$

We easily see that $\Delta_y , (x,y) = 0$. Furthermore,

$$\begin{aligned}
\left|\int_{\partial B_\epsilon} , (x,y)\nabla f(y)\cdot\nu(y)ds(y)\right| &\leq \sup|\nabla f(y)|\, Vol(\partial B_\epsilon)\frac{1}{\omega_{n-1}\epsilon^{n-2}} \\
&\leq \sup|\nabla f(y)|\,\epsilon^{n-1}\frac{1}{\epsilon^{n-2}} \\
&\leq \sup|\nabla f(y)|\,\epsilon
\end{aligned}$$

which goes to zero as $\epsilon \to 0$.

We now have to calculate $\nabla_y , (x,y)\cdot\nu(y)$. This is easily done since $\nu(y) = -|x-y|$ (the minus is because it is the outward pointing normal for $\mathbb{R}^n \setminus B_\epsilon$). Thus we really have $\frac{d}{dr}(\omega_{n-1}r^{2-n}) = (2-n)\omega_{n-1}r^{1-n}$ so $^- \int_\partial B_\epsilon f(y)\nabla_y , (x,y)\cdot\nu(y)ds(y) = \int_\partial B_\epsilon f(y)ds(y) \to f(x)$ as $\epsilon \to 0$. Thus if we let $\epsilon \to 0$ we get $\int_{\mathbb{R}^n} , (x,y)\Delta f(y)dy = f(x)$, or $\Delta_{\text{distr}(y)}, (x,y) = \delta_x(y)$.

Our second method uses the Fourier transform.

3. The Fundamental Solution. We now want to show that $u = Qf(x) = \int_{\mathbb{R}^n} , (x,y)f(y)dy$ is a solution to the equation $\Delta u = f$. First assume that f is C^∞, which implies that Qf is C^∞ (This is because it is a convolution). We first show that the equation is true in the sense of distributions. That is let $\phi \in C_c^\infty(\mathbb{R}^n)$ and show $\int_{\mathbb{R}^n} \Delta Qf(x)\phi(x)dx = \int_{\mathbb{R}^n} f(x)\phi(x)dx$. We do this as follows:

$$\begin{aligned}
\int_{\mathbb{R}^n} \Delta Qf(x)\phi(x)dx &= \int_{\mathbb{R}^n} Qf(x)\Delta\phi(x)dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} , (x,y)f(y)dy\Delta\phi(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ,\,(x,y)\Delta\phi(x)dxf(y)dy \\
&= \int_{\mathbb{R}^n} \phi(y)f(y)dy \\
&= \int_{\mathbb{R}^n} f(x)\phi(x)dx
\end{aligned}
$$

Now we have that Qf is a weak solution and is C^∞, so elliptic regularity tells us that it is an actual solution. Thus our Green's function is $G(x,y) = \,,(x,y)$.

Can we weaken the smoothness conditions?

4. The Green's Function for Δ_g on a Compact Manifold (M,g). We can now study the Green's function on a Riemannian manifold (M,g). Recall that this means that we would like to find a function $G(x,y)$ such that the function $Qf(x) = \int_M G(x,y)f(y)\mathrm{dV}(y)$ is the inverse of the Laplacian, i.e. $\Delta Qf(x) = f(x)$ for any $f \in L^2(M)$ and also that $Q\Delta u(x) = u(x)$ for all $u \in H_2(M)$. Unfortunately, this is not entirely plausible, since it would mean that $\Delta : H_2(M) \to L^2(M)$ is injective, which is not true since the constants are in the kernel (note that this is not true in the case $\mathbb{R}^n$ since the constants are not integrable). Thus we want

$$\Delta Qf(x) = f(x) - \frac{(f,1)}{(1,1)}1 = f(x) - \frac{1}{V}\int_M f(y)\mathrm{dV}(y)$$

where V is the volume of M, which means that $\Delta Qf(x)$ is the projection of $f(x)$ onto the orthogonal complement of the constant functions.

We first take a function $\eta(x,y) = \overline{\eta}(d_g(x,y))$ where d is the distance and $\overline{\eta}$ is a function with compact support within the injectivity radius and which is identically 1 in a neighborhood of zero. Let our first approximation be $H(x,y) = \omega_{n-1}d(x,y)^{2-n}\eta(x,y)$. Notice that H is symmetric in the two variables. We want to mimic our approach to $\mathbb{R}^n$, so let's compute $\Delta_{\mathrm{distr}(y)}H(x,y)$. First we will need some estimates.

5. Estimates on $\Delta_y H(x,y)$. We first derive the formula for the Laplacian of a radial function. Recall that in polar coordinates, we can write the metric as

$$g = dr^2 + r^{n-1}g_{ij}d\theta^i d\theta^j$$

We then compute in polar coordinates, letting $\sqrt{g} = \sqrt{det\,[g_{ij}]}$:

$$
\begin{aligned}
\Delta f(r) &= \frac{1}{r^{n-1}\sqrt{g}}\partial_i r^{n-1}\sqrt{g}g^{ij}\partial_j f(r) \\
&= \frac{1}{r^{n-1}\sqrt{g}}\partial_r r^{n-1}\sqrt{g}\partial_r f(r) \\
&= f''(r) + \frac{n-1}{r}f'(r) + f(r)\partial_r \log\sqrt{g}
\end{aligned}
$$

We now apply this to $H(x,y) = \frac{1}{(2-n)\omega_{n-1}}d(x,y)^{2-n}\eta(d(x,y))$. So $H(r) = \frac{1}{(n-2)\omega_{n-1}}r^{2-n}\eta(r)$,

$$\begin{aligned}\partial_r H(r) &= \partial_r \frac{1}{(n-2)\omega_{n-1}} r^{2-n}\eta(r)\\ &= \frac{1}{(2-n)\omega_{n-1}}\left[(2-n)r^{1-n}\eta(r)+r^{2-n}\eta'(r)\right]\end{aligned}$$

and

$$\partial_r^2 H(r) = \frac{1}{(2-n)\omega_{n-1}}\left[(2-n)(1-n)r^{-n}\eta(r)+(2-n)r^{1-n}\eta'(r)+(2-n)r^{1-n}\eta'(r)+r^{2-n}\eta''(r)\right]$$

so if you fix x, we find that find that, letting $r=d(x,y)$:

$$\begin{aligned}\Delta_y H(x,y) &= \Delta H(r)\\ &= \frac{1}{(2-n)\omega_{n-1}}\left[(2-n)(1-n)r^{-n}\eta(r)+(2-n)r^{1-n}\eta'(r)+(2-n)r^{1-n}\eta'(r)+r^{2-n}\eta''(r)\right]\\ &\quad+\frac{n-1}{r}\frac{1}{(2-n)\omega_{n-1}}\left[(2-n)r^{1-n}\eta(r)+r^{2-n}\eta'(r)\right]\\ &\quad+\frac{1}{(2-n)\omega_{n-1}}\partial_r \log\sqrt{g}r^{2-n}\eta(r)\\ &= \frac{\eta'(r)}{\omega_{n-1}}\left(2+\frac{n-1}{2-n}\right)r^{1-n}\\ &\quad \frac{1}{(2-n)\omega_{n-1}}\left(\eta''(r)+\partial_r\log\sqrt{g}\eta(r)\right)r^{2-n}\end{aligned}$$

6. Finding the Green's Function. We need

$$\begin{aligned}&\int_{M\setminus B_\epsilon}\mathrm{div}(\nabla_y H(x,y)\phi(y))\mathrm{dV}(y)-\int_{M\setminus B_\epsilon}\mathrm{div}(H(x,y)\nabla\phi(y))\mathrm{dV}(y)\\ &=\int_{\partial B_\epsilon}\nabla_y H(x,y)\cdot\nu(y)\phi(y)\mathrm{dV}(y)-\int_{\partial B_\epsilon}H(x,y)\nabla\phi(y)\cdot\nu(y)\mathrm{dV}(y)\end{aligned}$$

The left side is $\int_{M\setminus B_\epsilon}\Delta_y H(x,y)\phi(y)\mathrm{dV}(y)-\int_{M\setminus B_\epsilon}H(x,y)\Delta\phi(y))\mathrm{dV}(y)$. As for the right side, similar arguments (Do it!!!!!) to the above case show that it goes to $\phi(x)$ as $\epsilon\to 0$. Thus we find that

$$\int_M H(x,y)\Delta\phi(y))\mathrm{dV}(y)=\phi(x)+\int_M \Delta_y H(x,y)\phi(y)\mathrm{dV}(y)$$

or $\Delta_{\mathrm{distr}(y)}H(x,y)=\delta_x(y)+\Delta_y H(x,y)$.

Now, if $H(x,y)$ were the Green's function, then we would just form the fundamental solution $Q_1 f(x)=\int_M H(x,y)f(y)\mathrm{dV}(y)$. It is not, however, which we see by computing $\Delta Q_1 f(x)$. Again, let $\phi\in C_c^\infty(\mathbb{R}^n)$ and we compute:

$$
\begin{aligned}
\int_M Q_1 f(x)\Delta\phi(x)\mathrm{dV}(x) &= \int_M \int_M H(x,y)f(y)\mathrm{dV}(y)\Delta\phi(x)\mathrm{dV}(x) \\
&= \int_M \int_M H(x,y)\Delta\phi(x)\mathrm{dV}(x)f(y)\mathrm{dV}(y) \\
&= \int_M \left[\phi(y) + \int_M \Delta_x H(x,y)\phi(x)\mathrm{dV}(x)\right] f(y)\mathrm{dV}(y) \\
&= \int_M \left[f(x) + \int_M \Delta_x H(x,y)f(y)\mathrm{dV}(y)\right] \phi(x)\mathrm{dV}(x)
\end{aligned}
$$

So we get that $\Delta_{\text{distr}} Q_1 f(x) = f(x) + \int_M \Delta_x H(x,y)f(y)\mathrm{dV}(y)$.

Thus we need to understand the regularity of $\int_M \Delta_x H(x,y)f(y)\mathrm{dV}(y)$ and change our operator Q_1 to get the fundamental solution. We will try to find a new operator Q_2 so that $\Delta(Q_1 + Q_2) = f(x)$. To do this, we simply need to solve $\Delta u = -\int_M \Delta_x H(x,y)f(y)\mathrm{dV}(y)$ weakly.

Now, if we had that $f_2(x) = -\int_M \Delta_x H(x,y)f(y)\mathrm{dV}(y)$ is in $L^2(M)$ and that it integrates to zero, then we

Let's follow the same program we did before. We want to solve $\Delta u = f_2$. Let $Q_2 f(x) = \int_M H(x,y)f_2(y)\mathrm{dV}(y)$. We now check to see how close this is to the solution we want. By the last calculation we see that we get

$$
\begin{aligned}
\Delta_{\text{distr}} Q_2 f(x) &= f_2(x) + \int_M \Delta_x H(x,y)f_2(y)\mathrm{dV}(y) \\
&= f_2(x) - \int_M \Delta_x H(x,y) \int_M \Delta_y H(y,z)f(z)\mathrm{dV}(z)\mathrm{dV}(y) \\
&= f_2(x) - \int_M \int_M \Delta_x H(x,y)\Delta_y H(y,z)\mathrm{dV}(y)f(z)\mathrm{dV}(z)
\end{aligned}
$$

So we find that

$$
\Delta_{\text{distr}}(Q_1 + Q_2)f(x) = f(x) - \int_M \int_M \Delta_x H(x,y)\Delta_y H(y,z)\mathrm{dV}(y)f(z)\mathrm{dV}(z)
$$

We can, of course, continue this course of action indefinitely.

Now, we look at Q_2:

$$
\begin{aligned}
Q_2 f(x) &= \int_M H(x,y)f_2(y)\mathrm{dV}(y) \\
&= -\int_M H(x,y) \int_M \Delta_y H(y,z)f(z)\mathrm{dV}(z)\mathrm{dV}(y) \\
&= -\int_M \int_M H(x,y)\Delta_y H(y,z)\mathrm{dV}(y)f(z)\mathrm{dV}(z)
\end{aligned}
$$

Thus our second approximation to the Green's function is

$$
G_2(x,y) = H(x,y) - \int_M H(x,z)\Delta_z H(z,y)\mathrm{dV}(z)
$$

Now, if we could solve $\Delta_{\text{distr}(x)} F(x,y) = R_2 = \int_M \Delta_x H(x,z) \Delta_z H(z,y) \mathrm{dV}(z)$ where R_2 is continuous, then we would take $G(x,y) = G_2(x,y) + F(x,y)$ and we would be done. Unfortunately, R_2 is not necessarily continuous, so we continue until we become continuous using the following lemma.

Lemma 6.1. *Let $F(x,y) = \int_M G(x,z)H(z,y)\mathrm{dV}(z)$ and suppose that $|G(x,z)| \le \mathrm{Const} \cdot d(x,z)^{a-n}$ and $|H(z,y)| \le \mathrm{Const} \cdot d(z,y)^{b-n}$, where $0 < a, b < n$, then*

$$|F(x,y)| \le \left\{ \begin{array}{ll} \mathrm{Const} \cdot d(x,y)^{a+b-n} & \textit{if } a+b<n \\ \mathrm{Const} \cdot (1 + |\log d(x,y)|) & \textit{if } a+b=n \\ \mathrm{Const} & \textit{if } a+b>n \end{array} \right\}$$

Proof: Let $d = d(x,y)/2$. We now compute the integral in 3 parts:

$$\int_m = \int_{B_d(x)} + \int_{B_{3d}(y) \setminus B_d(x)} + \int_{M \setminus B_{3d}(y)}$$

Now we compute separately:

$$\begin{aligned} \left| \int_{B_d(x)} G(x,z)H(z,y)\mathrm{dV}(z) \right| &\le \int_{B_d(x)} |G(x,z)H(z,y)| \, \mathrm{dV}(z) \\ &\le \mathrm{Const} \int_{B_d(x)} d(x,z)^{a-n} d(z,y)^{b-n} \mathrm{dV}(z) \\ &\le \mathrm{Const} \cdot d^{b-n} \int_{B_d(x)} d(x,z)^{a-n} \mathrm{dV}(z) \\ &= \mathrm{Const} \cdot d^{b-n} \int_{S^{n-1}} \int_0^d r^{a-n} r^{n-1} dr d\theta \\ &= \mathrm{Const} \cdot d^{b-n} \mathrm{Vol}(S^{n-1}) \frac{1}{a} d^a \\ &= \mathrm{Const} \cdot d^{a+b-n} \end{aligned}$$

where the constant depends on a and n.

$$\begin{aligned} \left| \int_{B_{3d}(y) \setminus B_d(x)} G(x,z)H(z,y)\mathrm{dV}(z) \right| &\le \int_{B_{3d}(y) \setminus B_d(x)} |G(x,z)H(z,y)| \, \mathrm{dV}(z) \\ &\le \mathrm{Const} \int_{B_{3d}(y) \setminus B_d(x)} d(x,z)^{a-n} d(z,y)^{b-n} \mathrm{dV}(z) \\ &\le \mathrm{Const} \cdot d^{a-n} \int_{B_{3d}(y) \setminus B_d(x)} d(z,y)^{b-n} \mathrm{dV}(z) \\ &\le \mathrm{Const} \cdot d^{a-n} \int_{S^{n-1}} \int_0^{3d} r^{b-n} r^{n-1} dr d\theta \\ &= \mathrm{Const} \cdot d^{a-n} \mathrm{Vol}(S^{n-1}) \frac{1}{b} 3^b d^b \\ &= \mathrm{Const} \cdot d^{a+b-n} \end{aligned}$$

where the constant depends on b and n. And finally,

$$\begin{aligned}
\left|\int_{M\setminus B_{3d}(y)} G(x,z)H(z,y)\mathrm{dV}(z)\right| &\le \int_{M\setminus B_{3d}(y)} |G(x,z)H(z,y)|\,\mathrm{dV}(z) \\
&\le \text{Const} \int_{M\setminus B_{3d}(y)} d(x,z)^{a-n} d(z,y)^{b-n}\mathrm{dV}(z) \\
&\le \text{Const} \int_{M\setminus B_{3d}(y)} (d(z,y)-2d)^{a-n}\, d(z,y)^{b-n}\mathrm{dV}(z) \\
&= \text{Const} \int_{S^{n-1}} \int_{3d}^{K} (r-2d)^{a-n}\, r^{b-n} r^{n-1} dr d\theta \\
&\le \text{Const} \int_{3d}^{K} (r-2d)^{a+b-2n}\, r^{n-1} dr
\end{aligned}$$

Now, if we change variables to $s = r - 2d$ we get

$$\begin{aligned}
\int_{d}^{K-2d} s^{a+b-2n}(s+2d)^{n-1} ds &\le \int_{d}^{K-2d} s^{a+b-2n}\left[(2s)^{n-1} + (4d)^{n-1}\right] ds \\
&= \int_{d}^{K-2d} \left[s^{a+b-n-1} + (4d)^{n-1} s^{a+b-2n}\right] ds
\end{aligned}$$

The second term is

$$(4d)^{n-1}\frac{1}{a+b-2n+1}\left[(K-2d)^{a+b-2n+1} - d^{a+b-2n+1}\right]$$

where the constant depends on . The third inequality follows from the fact that $d(z,y) - 2d \le d(x,z)$.

□

7. Axiomatic approach to the Green's function. It may be easier to understand the derivation by showing the properties that we require of our Green's Function. We shall need the following:

Theorem 7.1. *If we can find a function $G(x,y)$ such that*

1. $\Delta_{\mathrm{distr}(y)} G(x,y) = \delta_x(y) - \frac{1}{V}$
2. $\Delta_y G(x,y) = 0$
3. $G(x,y) \sim d(x,y)^{2-n}$

Then $G(x,y)$ is the Green's Function for the Laplacian, i.e. if we define $Qf(x) = \int_M G(x,y)f(y)\mathrm{dV}(y)$ then

- $\Delta Qf(x) = f(x) - \frac{1}{V}\int_M f(y)\mathrm{dV}(y)$ *and*

- $Q\Delta f(x) = f(x) - \frac{1}{V}\int_M f(y)\mathrm{dV}(y)$

for appropriate f.

Proof: Let us just check to see if $Q\Delta f(x) = f(x) - \frac{1}{V}\int_M f(y)\mathrm{dV}(y)$. We first look weakly.

$$\begin{aligned}
(Q\Delta f(x), \phi(x)) &= \int Q\Delta f(x)\phi(x)\mathrm{dV}(y) \\
&= \int\int G(x,y)\Delta f(y)\mathrm{dV}(y)\phi(x)\mathrm{dV}(x) \\
&= \int \Delta_{\mathrm{distr}(y)} G(x,y) f(y)\mathrm{dV}(y)\phi(x)\mathrm{dV}(x) \\
&= \int \left(f(x) - \frac{1}{V}\int f(y)\mathrm{dV}(y)\right)\phi(x)\mathrm{dV}(x)
\end{aligned}$$

□

Proof: Although we have already done the proof, let's do it again for old time's sake. Let's first compute $\Delta_{\mathrm{distr}} Qf(x)$. We consider the divergence theorem on $M \setminus B_\epsilon(x)$:

$$\begin{aligned}
&\int_{M\setminus B_\epsilon(x)} \mathrm{div}_y(\nabla_y G(x,y)\phi(y))\mathrm{dV}(y) - \int_{M\setminus B_\epsilon(x)} \mathrm{div}_y(G(x,y)\nabla\phi(y))\mathrm{dV}(y) \\
&= \int_{\partial B_\epsilon(x)} \nabla_y G(x,y)\cdot\nu(y)\phi(y)\mathrm{dV}(y) - \int_{\partial B_\epsilon(x)} G(x,y)\nabla\phi(y)\cdot\nu(y)\mathrm{dV}(y)
\end{aligned}$$

and the left hand term is

$$\begin{aligned}
\int_{M\setminus B_\epsilon(x)} \Delta_y G(x,y)\phi(y)\mathrm{dV}(y) &- \int_{M\setminus B_\epsilon(x)} G(x,y)\Delta\phi(y)\mathrm{dV}(y) \\
&= -\int_{M\setminus B_\epsilon(x)} G(x,y)\Delta\phi(y)\mathrm{dV}(y)
\end{aligned}$$

because of property 2.

□