Preface

This book is devoted to the integrability and nonintegrability of systems of nonlinear differential equations. Differential equations and dynamical systems appear naturally in the description of many phenomena for which local processes are known. For instance, most physical laws, such as conservation of mass, energy, and momentum are *local* laws. The central problem is then to obtain *global* information on these phenomena. These elementary processes are typically nonlinear and, assuming continuity of the states of the system (the dependent variables) in time and space (the independent variables), their evolution is governed by *nonlinear* differential equations. For example, in classical physics, the gravitational forces between masses is nonlinear as are the electromagnetic interactions. In hydrodynamics, the nonlinearity of the Navier-Stokes equation comes from inertial effects. Also, autocatalytic chemical reactions are described by nonlinear differential equations through the mass-action law. These nonlinear effects give rise to complex structures whose complete description can be extremely difficult. Once the local equations are formulated in a particular context, the next problem is to "solve" these equations. Already, in this simple statement, there is an ambiguity. For the physicist, the applied mathematician or the chemist, to "solve" an equation means to obtain global information on the solution and, if possible, derive a closed-form solution for which the state of the dependent variables may be predicted for all given independent variables. In this sense, an equation can be solved if it can be locally represented by known functions. The mathematician, however, is often interested in a more fundamental problem related to the existence and uniqueness of the solutions, a prerequisite of any subsequent analytical approach.

The first attempt to solve differential equations either explicitly or by series expansions goes back to Euler, Newton, and Leibniz. The theory of integration for the equations of motion was subsequently expanded by the work of the analysts and mechanicians associated with the names of Lagrange, Poisson, Hamilton, and Liouville in the late 18th and 19th centuries. The basic idea underlying these works is that the solution can always be represented by the combination of known functions or by perturbation expansions. The notion of "integrability" was then introduced to describe the property of equations for which all local and global information can be obtained either explicitly from the solutions or implicitly from the constants of the motion.

In a reductionist approach, we can single out two different works that have radically changed the program of classical mechanics of the 19th century, and will serve as guidelines throughout this book. The first of these works is Kovalevskaya's study of the Euler equations for the motion of a rigid body with a fixed point. She used an ingenuous and innovative technique based on the behavior of the solution near the singularities in the complex plane to show that apart from the known integrable cases and a new one that she discovered, there is no other case for which the solution can be expressed exactly in terms of single-valued functions. In essence, she proved that within the class of single-valued functions, the general Euler equations are not integrable.

Second, Poincaré studied the existence of constants of motion for integrable Hamiltonians under small perturbations. He showed that, in general, there is no additional constant of motion other than the Hamiltonian itself which is analytic in the expansion parameter. That is, if a constant of motion exists for the unperturbed Hamiltonian, then, it cannot exist continuously as the perturbation parameter is increased. In his study of celestial mechanics, Poincaré also developed a geometric theory of solutions. His idea was to study asymptotic solutions as geometric sets which define the global qualitative behavior of solutions in the long time limit. Poincaré introduced the concept of homoclinic and heteroclinic orbits which connect fixed points to themselves and showed that perturbations of these orbits are the source of complex behaviors. He noticed that if the three-body problem could be solved, "the transcendents needed to solve it would differ radically form all the known ones" ("les transcendantes qu'il faudrait imaginer pour le résoudre diffèrent de toutes celles que nous connaissons") (Poincaré, 1899, p. 391). In many respects, Poincaré's investigations of complex motion were several decades ahead of their time. Indeed, his study was based on an entirely different approach, the analysis of the *qualitative* behavior of solutions. To determine the global behavior of the solution in the long-time limit, his idea was to exploit the topological properties of the solutions in phase space together with the analytical properties of the equation

Despite their differences, the approaches of Kovalevskaya and Poincaré share a common feature. The local analysis of the differential equation, close to its complex time singularities for Kovalevskaya and its phase space singularities for Poincaré, allows us to find global properties of the system. As a consequence of these works, both mathematicians and physicists shifted their interests away from the theory of integrability. Mathematicians realized that the essence of the qualitative theory of differential equations was based on the notion of *dynamical systems* for which the abstract formulation was laid down by Birkhoff. Physicists did not fully appreciate the importance of nonlinearities until the 1960's. With the seminal works of Lorenz (Lorenz, 1963), on the numerical evidence of chaotic motion, and of Hénon and Heiles, on a nonintegrable two-degrees of freedom Hamiltonian (Hénon & Heiles, 1964), dynamical systems theory radically changed the way scientists think about nonlinear problems.

The success of dynamical systems theory was so overwhelming that exact methods for integration were considered for years useless and non-generic. In particular, chaos theory shadowed the equally important discovery by Zabusky and Kruskal (1965) of solitons for the Korteweg-de Vries equation. Solitons are elementary solutions of partial differential equations that have simple interaction laws reminiscent of linear systems and they are considered as the hallmark of integrability in nonlinear partial differential equations. Soon after, many other integrable systems were discovered or rediscovered. Strikingly, for years, chaos, strange attractors, and ergodicity have been considered as the important features of dynamical systems with few degrees of freedom. However, solitons, pattern formation and ordered structures were the key features of systems with infinite degrees of freedom modeled by nonlinear partial differential equations. These conceptual differences emerging from nonlinear models seem shocking and show how crucial is the understanding of the phenomena of integrability and nonintegrability in dynamical systems.

To further define the problem, we distinguish "solvability" from "integrability". "Integrability" is an intrinsic property of a given system imposing strong constraints on the way solutions evolve in phase space whereas "solvability" is related to the existence of closed-form solutions. However, a universal definition of integrability for dynamical systems seems elusive. Clearly, it should at least be compatible with the intuitive notion of regular or irregular behaviors. For dynamical systems, irregular behavior is usually associated with bounded dynamics sensitive to initial conditions with neighboring trajectories diverging locally in phase space with local exponential rates measured by the Lyapunov exponents. These exponents cannot be computed in general and their numerical evaluation, involving long-time averages, may be an extremely arduous task. Therefore, integrability cannot be simply defined by the lack of irregular behavior since many nonintegrable systems have regular dynamics. Our modest standpoint for the understanding of the problem of integrability and nonintegrability in dynamical system is singularity analysis; that is, the analysis of differential equations in complex time. We will consider several definitions of integrability and show how they relate to the structure of the solution viewed as functions of a complex variable. Then, we will develop simple algorithmic methods to detect systems which lack the fundamental properties of integrability. With further assumptions on nonintegrable systems, we will explicitly relate nonintegrability to the existence of irregular behaviors and show that seemingly contradictory aspects of nonlinear systems can be understood within singularity analysis.

What this book is not about

Integrability and dynamical systems have become such important theories that they have acquired over the years different meanings for different people. In writing this book, unless necessary, I have tried to include subjects which were not already covered by other textbooks. I felt that I could not improve on the excellent exposition or more talented writers. To avoid any confusion, I would like to give a partial list of subjects (all of great interests) usually associated with the notion of integrability that are not (or only partially) covered here: Integrable systems (Perelomov, 1990; Audin, 1996), Hamiltonian systems (Goldstein, 1980; Marsden & Ratiu, 1994; Kozlov, 1998), Discrete systems (Grammaticos *et al.*, 1999), Soliton theory and Inverse scattering techniques (Ablowitz & Segur, 1981; Newell, 1985; Ablowitz & Clarkson, 1991), Lie group analysis (Olver, 1993; Ibragimov, 1999), Dynamical systems and Chaos (Guckenheimer & Holmes, 1983; Wiggins, 1988; Perko, 1996).

What this book is about

This is mainly a book of concepts and methods. I hope that by carefully defining some general concepts and by showing many illustrative examples, I have given the reader the tools to tackle her or his problem. I have tried to adopt the view that studying the integrability or nonintegrability of a given dynamical system is not Black Magic and that, in fact, a systematic approach can be followed. To many, integrability is a mysterious notion that appears to occur seemingly randomly in the study of dynamical systems. True enough, integrability is rare, but we should cherish and fully understand these rare instances. Ultimately, they are the key to a thorough understanding of regular and irregular behavior of dynamical systems.

In Chapter 1, I consider two simple dynamical systems and try to formulate simple questions on the behaviors of their solutions which can be answered by considering the integrability of the systems. Chapter 2 is a general introduction to vector fields and first integrals. I define different notions of integrability based on the existence of first integrals and explore elementary properties of Lax pairs. Chapter 3 is dedicated to singularity analysis and the Painlevé property. Integrability is defined through the behavior of solutions in complex time. Chapter 4 further explores the algebraic and analytic properties of a large class of vector fields for which explicit matrix representations can be given. In Chapter 5, I use the local properties of vector fields in both phase space and complex time to develop methods to prove the nonintegrability of a given system. Chapter 6 presents an elementary introduction to the theory of integrable and nonintegrable Hamiltonian systems. The results developed in Chapters 3 and 4 are adapted to the special nature of Hamiltonians. In Chapter 7, I consider integrable systems under perturbations and study the effect of the perturbation, both in complex time and in phase space.

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