Thanksgiving Take Home Quiz #13

Math 322. Fall, 2007.

1. Consider the initial value problem (IVP) defined by partial differential equation (PDE)

$$u_t = u_{xx} \qquad 0 \le x \le \pi, \quad t \ge 0 \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad u(\pi,t) = 0 \tag{2}$$

and the initial condition

$$u(x,0) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 \le x \le \pi \end{cases} . \tag{3}$$

   (a) Look for solutions of the PDE of the form $u(x,t) = F(x)G(t)$ and set up the corresponding eigenvalue problems (Hint: You should use the boundary conditions (2) to set up the eigenvalue problem for $F(x)$).

   (b) Solve the eigenvalue problem for $F(x)$ that you found in part a). Find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

   (c) Find the functions $G_n(t)$ corresponding to the eigenvalues λ_n that you found in part b), and write the solution $u_n = F_n(x)G_n(t)$ with no arbitrary constant.

   (d) Verify that the functions $u_n(x,t)$ that you found are indeed solutions of the PDE (1).

   (e) Write the general solution of the PDE as $u(x,t) = \sum_n A_n u_n(x,t)$ and find the constant coefficients A_n are chosen to satisfy the initial conditions (3).

2. Consider the boundary value problem (BVP) defined by Laplace's equation

$$u_{xx} + u_{yy} = 0 \qquad \text{on the square} \quad 0 < x, y < 10 \tag{4}$$

subject to the boundary conditions

$$u(0,y) = 0, \qquad u(10,y) = 0, \qquad u(x,10) = 0, \qquad u(x,0) = 100\sin(\pi x/10). \tag{5}$$

Solve the BVP using the method of separation of variables. That is:

   (a) Look for solutions of the PDE of the form $u(x,y) = F(x)G(y)$ and set up the corresponding eigenvalue problems (Hint: You should use the boundary conditions $u(0,y) = u(10,y) = 0$ to set up the eigenvalue problem for $F(x)$).

   (b) Solve the eigenvalue problem for $F(x)$ that you found in part a). Find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

   (c) Find the functions $G_n(y)$ corresponding to the eigenvalues λ_n that you found in part b). There are two arbitrary constants appearing in $G_n(y)$. Find a relationship between these constants by using the first boundary conditions $u(x,10) = 0$, (DO NOT TRY TO ENFORCE THE SECOND BOUNDARY CONDITION $u(x,0) = 100\sin(\pi x/10)$.). Write down explicit expressions for the solutions $u_n(x,y) = F_n(x)G_n(y)$ of the PDE.

   (d) Verify that the functions $u_n(x,y)$ that you found in part d) are indeed solutions of the PDE (4).

   (e) Write the general solution of the PDE as $u(x,y) = \sum_n A_n u_n(x,y)$ and find the constant coefficients A_n to satisfy the second boundary conditions $u(x,0) = 100\sin(\pi x/10)$).