ALGEBRAIC DEGENERACY AND PARTIAL INTEGRABILITY FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

A. GORIELY and L. BRENIG
Université Libre de Bruxelles, CP 231, Boulevard du Triomphe, 1050 Brussels, Belgium

Received 6 December 1989; accepted for publication 16 February 1990
Communicated by A.R. Bishop

We show that a certain type of representation of ordinary differential equations can be used together with nonlinear transformations to find integrability conditions and to construct the corresponding first integrals. These conditions are obtained through the study of the rank of one of the matrices introduced by the representation. We also point out that the existence of particular time-dependent first integrals is bijectively connected to the algebraic degeneracy of this matrix. Some examples connected with nonlinear sciences are analyzed and the comparison with other methods is presented.

1. A general representation of ODEs and quasimonomial transformations

In a recent paper [1] we have introduced the so-called "Lotka–Volterra representation" for systems of nonlinear polynomial differential equations:

$$\dot{x}_i=\lambda_i x_i+x_i\sum_{j=1}^{m}A_{ij}\prod_{k=1}^{n}x_k^{B_{jk}}\,,\quad i=1,\ldots,n\,,\quad m\geqslant n\,,\tag{1}$$

where A and B are rectangular real matrices (respectively $n\times m$ and $m\times n$). The x_i's are real functions of the time t.

Any system of ODEs which contains a finite number of polynomial terms can be written in this representation with an appropriate choice of matrices. Therefore, almost any physical system modeled by ODEs can be cast into form (1) and, as we shall see, their analysis can be made through the study of the fundamental matrices A and B.

Eq. (1) appears to be form-invariant under the "quasimonomial" transformations:

$$T_n(C):\quad x_i=\prod_{k=1}^{n}x_k'^{C_{ik}}\tag{2}$$

where C is an $n\times n$ invertible matrix.

Indeed, if we make the following assumptions

$$|x_i(t)|>0\quad\forall i\forall t\in[0,\infty]\,,\tag{3}$$

system (1) is mapped by transformation (2) on

$$\dot{x}_i'=\lambda_i' x_i'+x_i'\sum_{j=1}^{m}A_{ij}'\prod_{k=1}^{n}x_k'^{B_{jk}'}\,,\quad i=1,\ldots,n\,,\quad m\geqslant n\,,\tag{4}$$

with

$$\lambda_i'=(C^{-1}\lambda)_i\,,\tag{5a}$$

$$A_{ij}'=(C^{-1}A)_{ij}\,,\tag{5b}$$

$$B_{ij}'=(BC)_{ij}\,.\tag{5c}$$

If conditions (3) are not fulfilled, it is still possible (under certain restrictions) to keep the form-invariance by combining transformations (2) with translations on the x_i-variables [2].

In ref. [1], we have used representation (1) and transformations (2) in order to define *equivalence classes* among ODE systems. Each class is characterized by the matrix products BA and $B\lambda$ (which are invariant under $T_n(C)$, see eq. (5)). Furthermore, if matrices B and A are regular, there always exist universal *canonical forms* on which system (1) can be mapped. In the case of B regular (i.e. of rank n), one can obtain the first canonical form by mapping system (1) on an m-dimensional quadratic Lotka–Volterra system (in ref. [2], we have shown that this form can always be obtained, even if B is not a matrix of rank n). If matrix A is regular, a second canonical form can be reached by putting $C=A$ in $T_n(C)$. In this form, the system (4) contains at most one nonlinear term per equation.

A much more interesting situation arises when matrix A is not of rank n. The aim of this paper is to show that this corresponds to the existence of particular first integrals. In the sequel, we shall say that system (1) is *degenerate* if matrix A is not of maximal rank.

2. A is not of rank n

Let I be the following time-dependent first integral,

$$I=I(\{x\},t)=(x_1^{E_1}x_2^{E_2}\dots x_n^{E_n})\,e^{\rho t}, \tag{6}$$

where E_i and ρ are real parameters and at least one of the E_i is non-vanishing.

According to the name of the transformations (2), we shall call the first integral (6) a *"quasimonomial integral"*. We now establish the main result of this paper:

System (1) has $n-r$ independent quasimonomial first integrals, *if and only if* matrix A is of rank r ($r\leqslant n$).

2.1. Necessary condition

Let us first assume that matrix A is of rank r. We know from matrix algebra that there always exists an invertible matrix C such that

$$A'=C^{-1}A$$

$$=\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ A'_{n-r+1,1} & A'_{n-r+1,2} & \dots & A'_{n-r+1,m} \\ A'_{n-r+2,1} & A'_{n-r+2,2} & \dots & A'_{n-r+2,m} \\ \vdots & \vdots & & \vdots \\ A'_{n1} & A'_{n2} & \dots & A'_{n,m} \end{pmatrix}. \tag{7}$$

By applying on system (1) a transformation $T_n(C)$ with a matrix C satisfying (7), system (4) reads now

$$\dot{x}'_l=\lambda'_l x'_l\,,\quad l=1,\dots,n-r\,,$$

$$\dot{x}'_i=\lambda'_i x'_i+x'_i\sum_{j=1}^{m}A'_{ij}\prod_{k=1}^{n}x_k'^{B'_{jk}}\,,\quad i=n-r+1,\dots,n\,, \tag{8}$$

with B' and λ' defined in (5).

In this new set of variables, the $n-r$ first equations of system (8) are linear and can be integrated immediately:

$$x'_l=e^{\lambda'_l t}K_l\,, \tag{9}$$

where the K_l's are arbitrary constants.

In terms of the original variables, we obtain the $n-r$ quasimonomial integrals

$$x_1^{C_{l1}^{-1}}x_2^{C_{l2}^{-1}}\dots x_n^{C_{ln}^{-1}}=e^{\lambda'_l t}K_l\,,\quad l=1,\dots,n-r\,. \tag{10}$$

Actually, the $n-r$ vectors $(C_{l1}^{-1}, C_{l2}^{-1},\dots,C_{ln}^{-1})$ can be directly identified as the vectors defining the null space of matrix A. The research of integrability conditions as well as the construction of the corresponding integrals are thus reduced to matrix algebra. Indeed, the knowledge of matrix A alone is sufficient to find the first integrals (10). Practically, steps (8) and (9) can be skipped. The C matrix obtained from (7) can be inserted immediately into (10) to explicitly construct the $n-r$ integrals.

2.2. Sufficient condition

We now assume that system (1) has $n-r$ independent quasimonomial first integrals. These integrals can be written using a matrix E and a vector ρ,

$$x_1^{E_{11}}x_2^{E_{12}}\dots x_n^{E_{1n}}=K_1e^{\rho_1 t}\,,\quad x_1^{E_{21}}x_2^{E_{22}}\dots x_n^{E_{2n}}=K_2e^{\rho_2 t}\,,$$
$$\dots,\quad x_1^{E_{n-r,1}}x_2^{E_{n-r,2}}\dots x_n^{E_{n-r,n}}=K_{n-r}e^{\rho_{n-r}t}\,. \tag{11}$$

The independence of these integrals ensures matrix E to be regular. This allows us to apply to system (1) the transformation $T_n(C)$ defined by

$$C_{ij}^{-1}=E_{ij}\,,\quad i=1,\dots,n-r\,,\quad j=1,\dots,n\,,$$
$$=e_{ij}\,,\quad i=n-r+1,\dots,n\,,\quad j=1,\dots,n\,, \tag{12}$$

where the e_{ij}'s are choosen in such a way that the matrix C is invertible.

In the new variables, system (11) reads

$$x'_1=e^{\rho_1 t}K_1\,,\quad\dots,\quad x'_{n-r}=e^{\rho_{n-r}t}K_{n-r}\,. \tag{13}$$

These $n-r$ equations are solutions of the $n-r$ ODEs

$$\dot{x}'_i = \rho_i x'_i, \quad i=1, ..., n-r. \tag{14a}$$

In order to obtain an n-dimensional system equivalent to system (1), we complete the eqs. (14a) with r differential equations given by

$$\dot{x}'_i = \lambda'_i x'_i + x'_i \sum_{j=1}^{m} A'_{ij} \prod_{k=1}^{n} x'^{B'_{jk}}_k, \quad i=n-r+1, ..., n, \tag{14b}$$

with A' and B' defined by (5) and (12).

System (14) can be written in the form (4) with

$$A' = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ A'_{n-r+1,1} & \dots & A'_{n-r+1,m} \\ \vdots & & \vdots \\ A'_{n,1} & \dots & A'_{n,m} \end{pmatrix} \tag{15}$$

and $B' = BC$.

It is then obvious that matrix $A = CA'$ is of rank r. Let us stress that the ρ_i's are completely determined by the knowledge of E and λ:

$$\rho_i = \sum_{j=1}^{n} E_{ij} \lambda_j, \quad i=1, ..., n-r. \tag{16}$$

2.3. A short example: the three-dimensional generalized Lotka–Volterra equations

Let us consider system (1) with $m=n=3$. It has been shown that many physical systems can be cast in that form: three-dimensional Lotka–Volterra equations ($B=I$), Lamb equations for a three-mode operating cavity in a laser, the asymmetric top and resonant three-wave interaction equations (see ref. [3]).

If A is of rank 2 ($r=2$), there exists a quasimonomial integral:

$$x_1^{a_{31}} x_2^{a_{32}} x_3^{a_{33}} = \exp[(\lambda_1 a_{31} + \lambda_2 a_{32} + \lambda_3 a_{33})t] K_3, \tag{17}$$

where K_3 is an arbitrary constant and the a_{ij}'s denote the matrix of the cofactors of A.

Let us stress that there is, in this case, only one condition among the 8 independent parameters defining the system (the A_{ij}'s and λ_i's), the B_{ij}'s being fixed in a given physical model. This unique condition allows one to find a very general integral which is time-dependent and, typically, irrational. Despite its simplicity, only few theories provide integral (1) in its general form. For example, the singularity analysis [4] applied to a three-dimensional Lotka–Volterra system with 5 independent parameters provides integrals of the form (17) but with the restriction that a_{31}, a_{32} and a_{33} are rational numbers. Indeed, the Painlevé property is not satisfied in the general case since the E_i's are almost always irrational. The compatibility analysis [5] applied to the same particular case implies conditions between the λ_i's in order to obtain a time-independent integral. More recently, the Carleman embedding method has been applied to the n-dimensional Lotka–Volterra system. This analysis provides integral (17) in n dimensions and also other interesting invariants [6].

3. Complete degeneracy

The study of system (1) with a matrix A given by $A_{ij} = a_j$ provides another interesting situation. Indeed, in this case system (1) reads

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{j=1}^{m} a_j \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i=1, ..., n. \tag{18}$$

If $B=I$, system (18) reduces to the projective Riccati equations for which the general solution is known [7].

As a consequence of the degeneracy of A ($r=1$), we can construct $n-1$ integrals by choosing matrix C^{-1} as

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}. \tag{19}$$

According to section 1, the corresponding integrals are

$$\frac{x_i}{x_{i+1}} = K_i \exp[(\lambda_i - \lambda_{i+1})t], \quad i=1, ..., n-1. \tag{20}$$

Furthermore, the previous result can be generalized by adding a non-diagonal linear part to eq. (18):

$$\dot{x}_i = \sum_{j=1}^{n} L_{ij} x_i + x_i \sum_{j=1}^{m} a_j \prod_{k=1}^{n} x_k^{B_{jk}}, \quad i=1, \dots, n, \tag{21}$$

where the B_{ij}'s are positive integers.

First of all, we diagonalize the linear part by a linear transformation $y_i = \sum_{j=1}^{n} D_{ij} x_j$ which leads to

$$\dot{y}_i = \lambda_i y_i + y_i \sum_{j=1}^{m} a_j \prod_{k=1}^{n} (D^{-1}y)_k^{B_{jk}}, \tag{22}$$

which can be re-written in the form

$$\dot{y}_i = \lambda_i y_i + y_i \sum_{j=1}^{m'} a'_j \prod_{k=1}^{n} y_k^{B'_{jk}}, \tag{23}$$

where the λ_i's are the eigenvalues of matrix L (assuming that L has no degenerate eigenvalues).

In eq. (23) we have expanded the non-linear part in m' monomial terms. Obviously, system (23) is of type (18). This leads to the following $n-1$ integrals,

$$\frac{\sum_{j=1}^{n} D_{ij} x_j}{\sum_{j=1}^{n} D_{i+1,j} x_j} = K_i \exp[(\lambda_i - \lambda_{i+1})t],$$
$$i=1, \dots, n-1. \tag{24}$$

Furthermore, if the non-linear part is homogeneous, the remaining equation can be solved (it reduces to a Bernoulli equation).

4. Generalization

We propose here an extension of the previous results by considering systems which do not belong to class (1),

$$\dot{x}_i = \lambda_i x_i + g_i(\{x\}), \quad i=1, \dots, n, \tag{25}$$

where $\{g_i\}$ is a set of n analytical functions in $\mathbb{R}^n$.

For some type of systems (25), we find the same degeneracy we showed in section 3. Indeed, let us suppose that $g_i(\{x\})$ can be expanded in the following form,

$$g_i(\{x\}) = x_i \sum_{j=1}^{m} A_{ij} f_j(\{x\}), \tag{26}$$

where A_{ij} is an $n \times m$ matrix of rank $r \leqslant n$ and $\{f_j\}$ is another set of analytical functions in $\mathbb{R}^n$.

We now show that it is possible to map such a system onto a system with $n-r$ linear equations. Indeed, let N be a rectangular $(n-r) \times n$ matrix defined by the $n-r$ vectors of the null space of matrix A. A suitable matrix C^{-1} is given by

$$C_{ij}^{-1} = N_{ij}, \quad i=1, \dots, n-r, \quad j=1, \dots, n,$$
$$= M_{ij}, \quad i=n-r+1, \dots, n, \quad j=1, \dots, n, \tag{27}$$

where the M_{ij}'s are chosen such that C^{-1} is invertible. A transformation $T_n(C)$ maps (25) onto a decoupled system,

$$\dot{x}'_l = \lambda'_l x'_l, \quad l=1, \dots, n-r,$$
$$\dot{x}'_i = \lambda'_i x'_i + x'_i \sum_{j=1}^{m} (C^{-1}A)_{ij} f'_j(x'_l),$$
$$i=n-r+1, \dots, n, \tag{28}$$

where $f'_j(x'_l) = f_j(\prod_{k=1}^{n} x_k^{C_{kl}^{-1}})$.

Clearly, following the same lines of reasoning as above (steps (7) to (10)), we find the same quasimonomial integrals.

4.1. Example

Let us consider a system describing competition between populations [8],

$$\dot{x}_i = -\lambda x_i + (B_i - x_i) g(x_i) - x_i \sum_{k \neq i}^{n} g(x_k),$$
$$i=1, \dots, n, \tag{29}$$

where $\lambda \geqslant 0$, $B_i \geqslant 0$ and g is an analytical function.

This system has been studied in various fields (neural nets, psychophysiology, ecology, etc., see ref. [8]). The function $g(x_i)$ is often well modeled by a sigmoid. System (28) can be cast in form (25) and (26) with

$$A = \begin{pmatrix} B_1 & & & 0 & -1 & -1 & \dots & -1 \\ & B_2 & & & -1 & -1 & \dots & -1 \\ & & \ddots & & \vdots & \vdots & & \vdots \\ 0 & & & B_n & -1 & -1 & \dots & -1 \end{pmatrix}, \tag{30}$$

where A is an $n \times 2n$ matrix, and

$$f_i = g(x_i)/x_i\,, \quad i=1,...,n\,,$$
$$= g(x_i), \qquad i=n+1,...,2n\,. \tag{31}$$

Clearly, if $r+1$ components B_i vanish, matrix A is of rank r. It is then easy to deduce from our procedure $n-r$ quasimonomial integrals.

5. Concluding remarks

Let us remark that in both cases (sections 2 and 4), the integrals found by our approach do not depend on the degree of non-linearity of the vector field (i.e. the matrix B in section 2 and the functional forms of the f_i's in section 4). They rather depend on the algebraic degeneracy of matrix A.

The existence of such quasimonomial integrals allows one to divide $\mathbb{R}^n$ into invariant regions (defined by the coordinate planes for integrals (10), or more general surfaces for the case discussed in section 3) which are mapped onto themselves under the evolution of the dynamical system.

The quasimonomial formalism seems to be well adapted for the search of integrability conditions as well as for the construction of first integrals. The method reported in this paper is only one aspect of the use of these transformations in the field of integrability. Other methods have already been shown and further developments are now being studied.

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