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A Mel'nikov vector for N-dimensional mappings

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Abstract

We use the Fredholm alternative to derive a Mel'nikov vector for perturbations of N-dimensional maps with homoclinic connections. If the unperturbed mapping is integrable, this vector assumes a simple form, which we use to determine conditions for transversal and tangential intersection between the invariant manifolds in a four-dimensional map of the McMillan type. We also discuss conditions for non-transversal intersection which accurately predict the crossing of invariant manifolds from one part of 4-D space into another.

1. Introduction

It is well-known that the Mel'nikov function (or integral) has played an important role in determining approximately the onset of chaos via homo- (or hetero-)clinic tangencies in periodically perturbed two-dimensional flows (see Refs. [1,2], and references therein). However, despite a number of interesting results [2–4], the extensions of Mel'nikov's theory to perturbations of higher dimensional systems was restricted for a long time to specific examples and did not offer convenient tools to treat the more general case.

One of the reasons for this limitation has been the fact that Mel'nikov's approach is primarily *geometrical* and relies heavily on the particular features of two-dimensional Poincaré maps. As soon as one considers the higher dimensional case, serious difficulties arise with defining distances between invariant manifolds and conditions under which these manifolds intersect.

In a recent paper [5], Chow and Yamashita demonstrated how to avoid these difficulties altogether, by formulating the problem in terms of the existence of *bounded solutions* of a system of *N linear* equations. In particular, using techniques developed earlier in Refs. [6] and [7], they applied the Fredholm alternative to the variational equations about the (known) separatrix solution of the unperturbed problem and derived a Mel'nikov vector yielding explicit conditions for the existence of homo- (or hetero-)clinic orbits of the perturbed system.

In this Letter, we apply Chow and Yamashita's approach and derive a Mel'nikov vector for perturbations of N-dimensional mappings,

$$\boldsymbol{x}_{n+1} = \boldsymbol{F}(\boldsymbol{x}_n) = \epsilon \boldsymbol{G}(\boldsymbol{x}_n), \qquad (1)$$

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$n = 0, 1, 2, \ldots,$ with $F, G: \mathbb{R}^N \to \mathbb{R}^N$, which for $\epsilon = 0$ possess a known homo- (or hetero-)clinic orbit. In the $N = 2$ case, our result coincides with the Mel'nikov function of a two-dimensional mapping [8], obtained by geometric arguments.

As demonstrated in Ref. [5], if the $\epsilon = 0$ case is Hamiltonian, the Mel'nikov vector assumes a more convenient form, since its components are related to the gradients of the m integrals of the $N\,(=2m)$-dimensional unperturbed problem. Here, we apply this theory to perturbations of the four-dimensional symplectic mapping

$$F(x_n) = \begin{pmatrix} -u_n + 2Kx_n/(x_n^2 + y_n^2 + 1) \\ x_n \\ -v_n + 2Ky_n/(x_n^2 + y_n^2 + 1) \\ y_n \end{pmatrix}, \tag{2}$$

with $x_n = (x_n, u_n, y_n, v_n)$. This map is of the McMillan type [8] and is known to be integrable [9] with analytic invariants

$$I_1 = x_n^2 + u_n^2 + y_n^2 + v_n^2 + (x_n^2 + y_n^2)(u_n^2 + v_n^2) - 2K(x_n u_n + y_n v_n), \tag{3a}$$

$$I_2 = x_n v_n - y_n u_n. \tag{3b}$$

It has physical significance in that it yields the static envelope solutions of the Ablowitz–Ladik integrable discretization of the nonlinear Schrödinger equation [10].

For $K > 1$, the origin is a saddle point, with two-dimensional stable and unstable manifolds, on which the solution $x_n^{(0)}(s)$ of (1) with $\epsilon = 0$ is known explicitly, as a function of $s \in \mathbb{R}^2$. We define a Mel'nikov vector, $M(s)$, with components

$$M_i(s) = \sum_{n=-\infty}^{\infty} \tilde{q}_{n+1}^{(i)} G(x_n^{(0)}(s)), \qquad i = 1, 2, \tag{4}$$

where $\tilde{q}_n^{(i)}$ is the transpose of the ith bounded solution of the adjoint (linear) variational problem about $x_n^{(0)}$. Now, if for some $s = s^*$, it happens that

$$M(s^*) = 0, \tag{5a}$$

this offers a necessary condition for the existence of bounded solutions for all n, hence homoclinic orbits of the perturbed map (1). Furthermore, if in addition to (5a) [4]

$$\det DM(s^*) \neq 0 \tag{5b}$$

(DM being the Jacobian of M) these orbits would correspond to transversal intersections of the invariant manifolds and the occurrence of chaos near the saddle point at $\mathbf{0}$.

Applying this criterion to the map (1) with F given by (2) and

$$G(x_n) = \begin{pmatrix} \gamma_1 x_n + \delta_1 u_n \\ 0 \\ \gamma_2 y_n + \delta_2 v_n \\ 0 \end{pmatrix}, \tag{6}$$

we first show, for δ_1, δ_2 small, that there exist $s = s^*$, for many γ_1, γ_2 values, such that (5) holds and transversal intersections occur. We then demonstrate how, for certain (γ_i, δ_i), $i = 1, 2$, one can satisfy (5a), $\det DM(s^*) = 0$ and an additional non-degeneracy criterion on the partial derivatives of M to obtain conditions for tangency between the invariant manifolds of the saddle point.

Interestingly enough, in the case of our example (6), we find that such conditions of non-transversal intersections can only be satisfied at points where $M_1(s_1^*)=0$, $M_2(s_2^*)=0$), with $s_1^* \neq s_2^*$. Still, we obtain predictions for the invariant manifolds crossing from one large region of phase space into another, which agree well with numerical computations, for $\gamma_{1,2} \leqslant 0.1$ and $\delta_{1,2} \leqslant 0.05$ ($\epsilon = 1$).

2. The Mel'nikov vector for maps

Consider an N-dimensional mapping of the form (1), which possesses, for $\epsilon = 0$, a saddle point at $\mathbf{0}$ with homoclinic connection, i.e. with orbits

$$x_n^{(0)}(s) \to 0, \qquad n \to \pm\infty, \tag{7}$$

parametrized by an m-dimensional variable $s \in \mathbb{R}^m$, $m < N$. Assume for simplicity that $\mathbf{0}$ remains a saddle point, for $\epsilon \neq 0$, as well ($|\epsilon| \ll 1$). We want to derive conditions for the perturbed system to possess such bounded orbits.

Let us consider the variational problem about this homoclinic orbit, (7), of the unperturbed system

$$q_{n+1} = D_n q_n, \qquad D_n = DF(x_n^{(0)}), \tag{8}$$

D_n being the Jacobian of F, $q_n \in \mathbb{R}^N$, and introduce the adjoint variational equation

$$\tilde{q}_{n+1} = \tilde{q}_n D_n^{-1}, \tag{9}$$

where the $\tilde{q}_n$ are N-dimensional row vectors.

The fundamental solution matrix Q_n ($\tilde{Q}_n$) of Eq. (8) (Eq. (9)) is an invertible matrix whose columns (rows) are linearly independent solutions of the variational (adjoint) equations

$$Q_{n+1} = D_n Q_n, \qquad \tilde{Q}_{n+1} = \tilde{Q}_n D_n^{-1}. \tag{10}$$

Note that there exists a simple relation between these fundamental solutions,

$$\tilde{Q}_{n+1} Q_{n+1} = \tilde{Q}_n Q_n = \tilde{Q}_0 Q_0 = \text{const}. \tag{11}$$

Now, consider a solution x_n of the perturbed system and assume that it can be expanded as a power series in ϵ,

$$x_n = x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)} + \ldots \,. \tag{12}$$

We then obtain from (12) and (1), to first order in ϵ, the variational equation about this solution,

$$x_{n+1}^{(1)} = D_n x_n^{(1)} + G(x_n^{(0)}), \tag{13}$$

cf. (8). According to the ideas developed in Ref. [5] (see also Refs. [11,12]), a necessary condition that $x_n^{(1)}$ (and hence x_n to $\mathcal{O}(\epsilon)$) remain bounded, for all integers n, can be expressed by the Fredholm alternative as follows:

Let $\tilde{q}_n$ be a bounded solution of the adjoint variational system (9) and multiply Eq. (13) by $\tilde{q}_{n+1}$, summing over all n, to obtain

$$\sum_{n=-\infty}^{\infty} \left(\tilde{q}_{n+1} x_{n+1}^{(1)} - \tilde{q}_{n+1} D_n x_n^{(1)} \right) = \sum_{n=-\infty}^{\infty} \tilde{q}_{n+1} G(x_n^{(0)}(s)). \tag{14}$$

It is clear from (14) that, if for some $s = s^*$,

$$\sum_{n=-\infty}^{\infty} \tilde{q}_{n+1} G(x_n^{(0)}(s^*)) = 0, \tag{15}$$

this provides a necessary condition for $x_n^{(1)}$ to be bounded for all n, as it implies that the sum on the l.h.s of (14) converges to

$$\sum_{n=-\infty}^{\infty} \left(\tilde{q}_{n+1} x_{n+1}^{(1)} - \tilde{q}_n x_n^{(1)} \right) = 0, \tag{15'}$$

cf. (9). On the other hand, if the $x_n^{(1)}$ are bounded, the series (15′) must converge, since the $\tilde{q}_n$ in general vanish exponentially, as $n \to \pm\infty$ (see also Section 3, below).

Let us now assume that the separatrix solution $x_n^{(0)}(s)$ is parametrized by $s \in \mathbb{R}^m$, cf. (7). We may, therefore, use (15) to define a Mel'nikov vector, $M(s)$, with components

$$M_i(s) = \sum_{n=-\infty}^{\infty} \tilde{q}_{n-1}^{(1)} G(x_n^{(0)}(s)), \qquad i = 1, 2, \ldots, m, \tag{16}$$

where m is the dimension of the bounded subspace of solutions of Eqs. (9) and (8). Thus, by analogy with the Mel'nikov vector for flows, we propose, as a necessary condition for the transversal intersection of invariant manifolds, the existence of *simple zeros* of $M(s)$, i.e. $s = s^*$ values, where

$$M_i(s^*) = 0, \qquad i = 1, 2, \ldots, m, \tag{17}$$

with [4]

$$\det DM(s^*) \neq 0. \tag{18}$$

If, on the other hand, condition (18) is replaced by an *equality* and an additional condition on the derivatives of M holds [5] (see below), we would expect (17), (18) to imply the occurrence of homoclinic tangencies which is an important precursor of the onset of chaos, near the saddle point at $\mathbf{0}$.

We remark that, in the two-dimensional example studied in Ref. [8], the Mel'nikov function (16) (with $m = 1$ and $s \in \mathbb{R}$) is identical to the one obtained by (completely different) geometrical arguments.

3. Invariant manifold intersections in a four-dimensional map

Let us now apply the above criterion to a four-dimensional (4-D) map (1), with F, G given by (2) and (3). Note that the unperturbed ($\epsilon = 0$) map is symplectic and possesses two analytic integrals given by (3). For $K > 1$, the origin is a saddle point with two-dimensional stable and unstable manifolds, which join smoothly, at $\epsilon = 0$. On these manifolds, the exact homoclinic solution $x^{(0)}(s)$, cf. (7), of the unperturbed equations is

$$x_n^{(0)}(s) = x_n^{(0)}(t, \theta) = \begin{pmatrix} a(\theta)\,\mathrm{sech}(nw + t) \\ a(\theta)\,\mathrm{sech}[(n-1)w + t] \\ b(\theta)\,\mathrm{sech}(nw + t) \\ b(\theta)\,\mathrm{sech}[(n-1)w + t] \end{pmatrix}, \tag{19}$$

with t, θ real arbitrary parameters and

$$a = \sinh w \cos\theta, \qquad b = \sinh w \sin\theta, \qquad w \equiv \cosh^{-1} K. \tag{19'}$$

Now, from the knowledge of (19), we can obtain immediately a solution of the variational equation (8) by differentiating (19) with respect to t,

$$q_n^{(1)} = \frac{\partial x_n^{(0)}(t, \theta)}{\partial t}. \tag{20}$$

Using now Eqs. (8)–(11), we easily find one (bounded) solution of the adjoint equations, as the row vector orthogonal to (20),

$$\tilde{q}_n^{(1)} = (-aS_{n-1}T_{n-1}, aS_nT_n, -bS_{n-1}T_{n-1}, bS_nT_n), \tag{21}$$

where

$$S_n \equiv \mathrm{sech}(nw + t), \qquad T_n \equiv \tanh(nw + t). \tag{22}$$

It is easy to check that, as in the theory for flows [5], here also, (21) is equal to the gradient of one of the integrals of the $\epsilon = 0$ system: $\tilde{q}_n^{(1)} = \nabla I_1(x_n^{(0)})$.

More generally, let us show that the gradient of an integral yields a solution of the adjoint variational equations. Indeed, for $\epsilon = 0$ one has

$$\frac{\partial I_i(x_n)}{\partial x_n} = \nabla I_i(x_n) = \nabla I_i(x_{n+1}) = \frac{\partial I_i(F(x_n))}{\partial F} DF(x_n) = \frac{\partial I_i(x_{n+1})}{\partial x_{n+1}} D_n, \tag{23}$$

cf. (8). We see, therefore, from (23), that

$$\tilde{q}_n^{(i)} = \nabla I_i(x_n), \quad i = 1, 2, \dots, m, \tag{24}$$

satisfies (9) and, hence, due to the independence and boundedness of the integrals, different integrals provide independent solutions of the adjoint problem and (24) gives all bounded solutions of the adjoint equations. Thus, we determine the second bounded solution of Eq. (9), using the second integral of the unperturbed system, (3b),

$$\tilde{q}_n^{(2)} = \nabla I_2(x_n^{(0)}) = (bS_{n-1}, -bS_n, -aS_{n-1}, aS_n), \tag{25}$$

cf. (19), (22).

Substituting now (21), (25) and (6) into our formula (16), we find the two components of the Mel'nikov vector of our problem,

$$M_1(t, \theta) = -\sum_{n=-\infty}^{\infty} S_n T_n \left[(a^2\gamma_1 + b^2\gamma_2) S_n + (a^2\delta_1 + b^2\delta_2) S_{n-1} \right], \tag{26a}$$

$$M_2(t, \theta) = ab \sum_{n=-\infty}^{\infty} \left[S_n^2(\gamma_1 - \gamma_2) + S_n S_{n-1}(\delta_1 - \delta_2) \right]. \tag{26b}$$

These $M_i(t, \theta)$ are actually periodic in t (with the same period w) and can be expressed in terms of Jacobi elliptic functions, using formulas given in Ref. [8]. They are also π-periodic in θ, as is evident from (19′).

It is quite convenient to evaluate numerically these (rapidly convergent) sums (26), plot them in 3-D space as functions of t and θ and locate graphically the desired points $s^* = (t^*, \theta^*)$ at which

$$M_1(t^*, \theta^*) = M_2(t^*, \theta^*) = 0, \tag{27}$$

by looking at intersection points between the nodelines of M_1 and M_2 in the s-plane.

In Fig. 1, we plot the two components of the Mel'nikov vector, as well as their nodelines, showing the existence of two points, where (27) holds. Hence, since at these points

$$\frac{\partial M_1(s^*)}{\partial t} \neq 0, \qquad \frac{\partial M_2(s^*)}{\partial \theta} \neq 0, \tag{28}$$

we conclude that (at these parameter values) condition (18) is satisfied and the invariant manifolds of the saddle point at $\mathbf{0}$ intersect transversely. We would therefore expect that these points correspond to isolated homoclinic orbits of the system.

Let us now concentrate on the case of tangential intersection of W^u and W^s, where we have according to Ref. [5], the following sufficient conditions,

$$M_i(s_i^*) = \partial_t M_i(s_i^*) = 0, \qquad i = 1, 2, \tag{29}$$

for some $s_i^* = (t_i^*, \theta_i^*) \in \mathbb{R}^2$ on the manifolds, and

$$\text{rank} \begin{pmatrix} \partial_t M_1(s_1^*) & \partial_t M_2(s_2^*) & \partial_\mu M_1(s_1^*) & \partial_\mu M_2(s_2^*) \\ \partial_t^2 M_1(s_1^*) & \partial_t^2 M_2(s_2^*) & \partial_{t\mu}^2 M_1(s_1^*) & \partial_{t\mu}^2 M_2(s_2^*) \end{pmatrix} = 4, \tag{30}$$

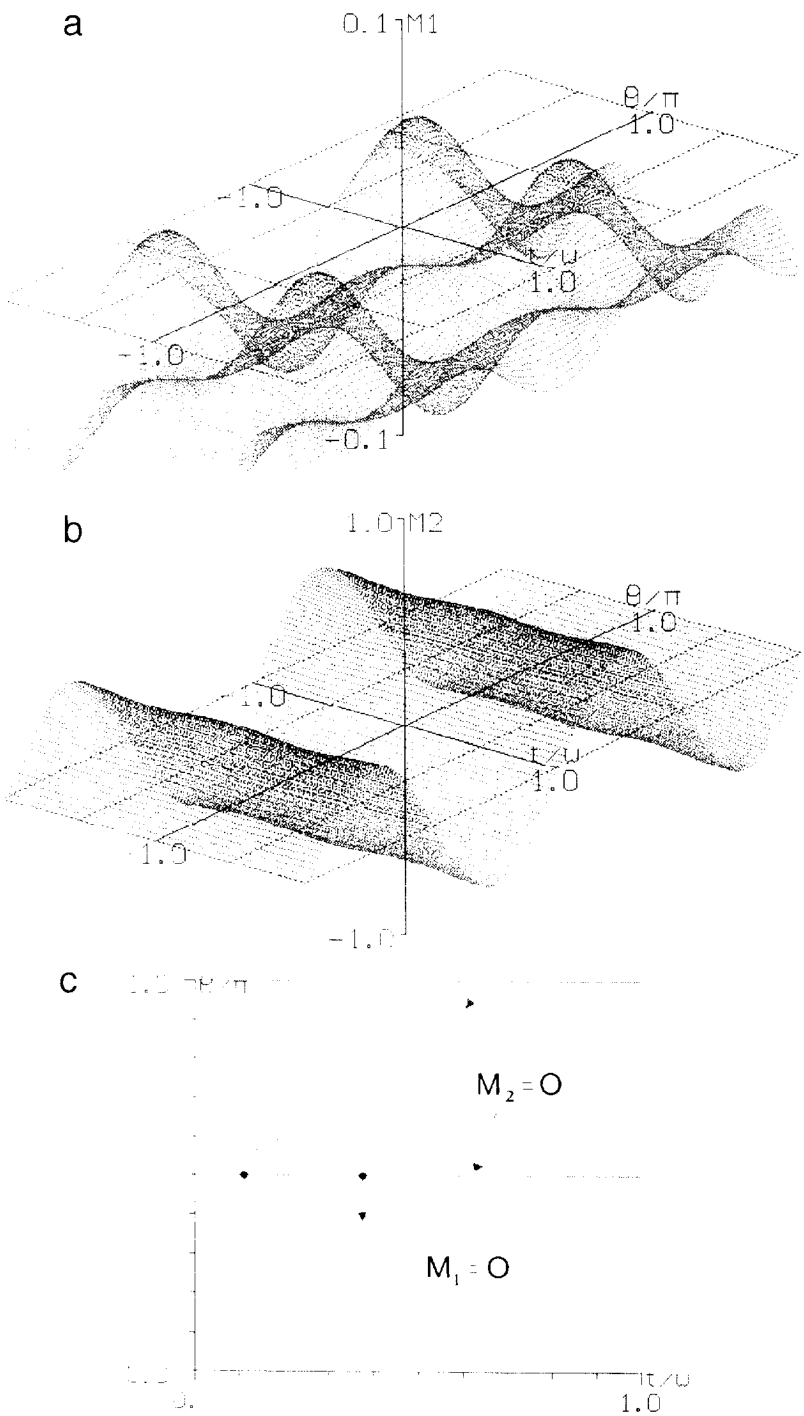


Fig. 1. The bounded components of the Mel'nikov vector (a) $M_1(t, \theta)$, (b) $M_2(t, \theta)$ and (c) their nodelines in the $(t/w, \theta/\pi)$, intersecting at two points. Here, $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\gamma_1 = 0.06$, $\gamma_2 = -0.06$.

with $\mu = \gamma_1, \gamma_2, \delta_1, \delta_2$. Note that in Ref. [5] $s_1^* = s_2^*$ is required. Here, however, we do not restrict ourselves to this constraint and allow the components of $\boldsymbol{M}$ to satisfy the tangency conditions (29) and (30) at distinct points on the manifolds.

For the Mel'nikov function (26), Eqs. (27) have two classes of solutions those that hold on isolated homoclinic points for arbitrary θ, and those that annihilate M_i and $\partial_t M_i$ identically along entire orbits for certain θ. The latter do not satisfy (30) and have to be discarded. The first class defines certain surfaces in parameter space $(\gamma_1, \gamma_2, \delta_1, \delta_2)$ as follows [13]. From $M_1 = \partial_t M_1 = 0$ we find [13]

$$\gamma_i = \pm \delta_i c(t_1^*), \qquad i = 1, 2, \tag{31a}$$

while $M_2 = \partial_t M_2 = 0$ leads to

$$\gamma_1 - \gamma_2 = (\delta_1 - \delta_2) d_i(t_2^*), \qquad i = 1, 2. \tag{31b}$$

A section at constant δ_i is displayed in Fig. 2a. The constants $\boldsymbol{c}$ and $\boldsymbol{d}_i$ are given in terms of periodic functions of S_n and T_n evaluated at the points of intersection t_i^* which are determined uniquely (mod w) from the tangency conditions. It is interesting that the t_1^* and t_2^* values we find are distinct, meaning that a true homoclinic tangency, at a single point s^*, as defined in Ref. [5], does not exist in our example. Still, non-transversal intersections are predicted by the above results at parameter values which correspond to the intersections of the surfaces (31) in γ_i, δ_i space (see points A, B in Fig. 2a).

To study the validity of these predictions we have developed a numerical method which gives parameter values for homoclinic tangency accurately and independently from the Mel'nikov approach: We compute, for $\epsilon \neq 0$, the eigenvectors of the linearized equations at $\mathbf{0}$, approximate $\boldsymbol{W}^{\mathrm{u}}$ and $\boldsymbol{W}^{\mathrm{s}}$ by their corresponding planar eigenspaces and place on a parallelogram, defined by the eigenvectors of $\boldsymbol{W}^{\mathrm{u}}$ with positive coordinates, a grid of 20×20 points very close to $\mathbf{0}$. We then "map out" this $\boldsymbol{W}^{\mathrm{u}}$ in four dimensions, by iterating these 400 points forward, a few times, checking whether any of them cross the corresponding positive part of $\boldsymbol{W}^{\mathrm{s}}$, upon their return near $\mathbf{0}$. If $\boldsymbol{W}^{\mathrm{s}}$ and $\boldsymbol{W}^{\mathrm{u}}$ are to intersect at some points in $\mathbb{R}^4$ they will also do so in the neighborhoud of the origin. Thus, it is sufficient to check whether $\boldsymbol{W}^{\mathrm{u}}$ intersects the planar approximation of $\boldsymbol{W}^{\mathrm{s}}$ near $\mathbf{0}$.

This can be studied by observing whether, after a number of iterations, some points on $\boldsymbol{W}^{\mathrm{u}}$ have projections in the $x_n < 0$ and/or $y_n < 0$ subspaces. For fixed δ_1, δ_2 this divides the γ_1, γ_2 plane in four regions, labeled I–IV, in Fig. 2b, as follows.

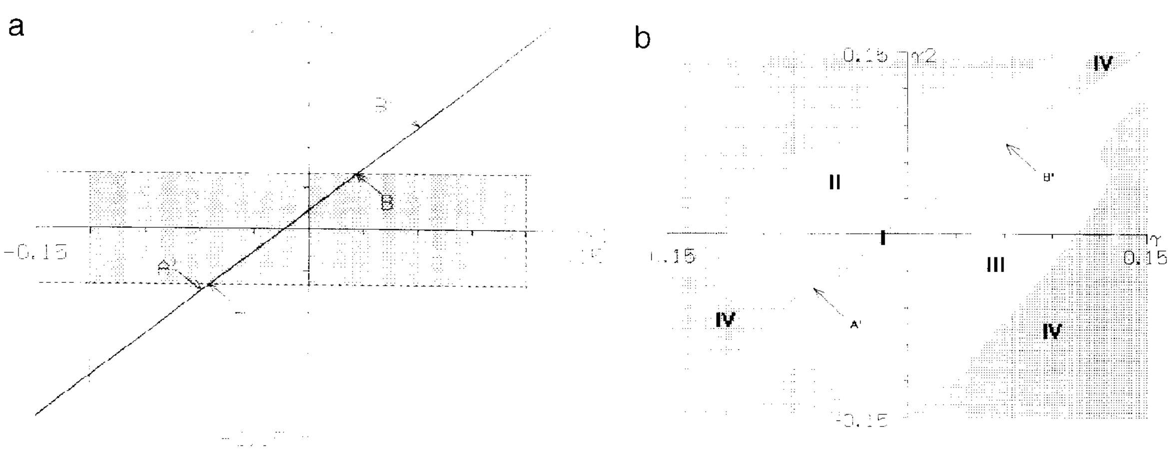


Fig. 2. (a) Vertical and horizontal lines, determining the central shaded parallelogram where $M_1 < 0$ (cf. (31a)) and diagonal lines (forming a very thin band in this figure, cf. (31b)) for $\delta_1 = 0.03$, $\delta_2 = 0.01$. Non-transversal intersections in 4-D are predicted at the nearest intersections of these lines to $\gamma_1 = \gamma_2 = 0$, i.e at the points A, B. (b) Numerical experiments show that the actual location of these points is at A', B'. Regions I, II, III and IV are as described in the text.

Region I. No intersections, all W^u points are trapped in the $x_n \geqslant 0$, $y_n \geqslant 0$ quadrant (see Fig. 3a).
Region II. Intersections in the x_n–u_n projections, trapping in the $y_n > 0$ subspace (see Fig. 3b).
Region III. Intersections in the y_n–v_n projections, trapping in the $x_n > 0$ subspace (see Fig. 3c).
Region IV. Intersections in both projections. Parts of W^u extend over all 4-D space (see Fig. 3d).

Thus, it becomes clear from these results that tangential intersections in 4-D are expected at parameter values where the above regions I–IV meet, marked by A′ and B′ in Fig. 2. Taking, for example, $\delta_1 = 0.03$, $\delta_2 = 0.01$, one of these points is $A' = (\gamma_1^{num}, \gamma_2^{num}) = (-0.0588, -0.0447)$. For the same case, our Mel'nikov theory gives $A = (\gamma_1^{th}, \gamma_2^{th}) = (-0.0549, -0.0403)$.

In a similar way, we determine for several choices of δ_1 and δ_2, γ_i^{num} values at which W^u intersects non-transversally W^s, and list them in Table 1, together with the corresponding theoretical values γ_i^{th}, predicted by the Mel'nikov vector analysis.

Note that the agreement between numerical and analytical results is quite good for the $\gamma_i < 0$ points A and A′. By comparison, the corresponding agreement between the $\gamma_i > 0$ points B and B′ becomes increasingly poorer, the more the δ_i values differ from each other (see Table 1). This might be explained by the fact that, for $\gamma_i > 0$, the x_n, y_n terms in (6) add "constructively" to the corresponding u_n, v_n terms (since $\delta_1 > \delta_2$), whereas, for $\gamma_i < 0$, the cumulative magnitude of these perturbations is considerably smaller, which may account for the better agreement between theory and practice, in that case.

4. Discussion

Using the Fredholm alternative for the existence of bounded solutions of a linear variational problem, we have derived in this paper, a Mel'nikov vector for ϵ-perturbations of N-dimensional maps, which possess, for $\epsilon = 0$, a smooth homoclinic connection. Our work is analogous that of Chow and Yamashita [5] for periodic perturbation of N-dimensional flows.

In case the unperturbed N (= 2m)-dimensional map has m integrals, $I_i(\boldsymbol{x}_n) = \text{const}$, we showed that the Mel'nikov vector $\boldsymbol{M}(s)$ has m components given in terms of the solutions $\tilde{\boldsymbol{q}}_n^{(i)} = \nabla I_i(\boldsymbol{x}_n^{(0)}(s))$ of the adjoint equations of the variational problem about the separatrix solution $\boldsymbol{x}_n^{(0)}(s)$, $s \in \mathbb{R}^m$, of the $\epsilon = 0$ system.

Thus, one can write down necessary conditions for transversal intersection between the corresponding invariant manifolds of the perturbed ($\epsilon \neq 0$) system, by requiring that the Mel'nikov vector possess simple zeros, i.e.

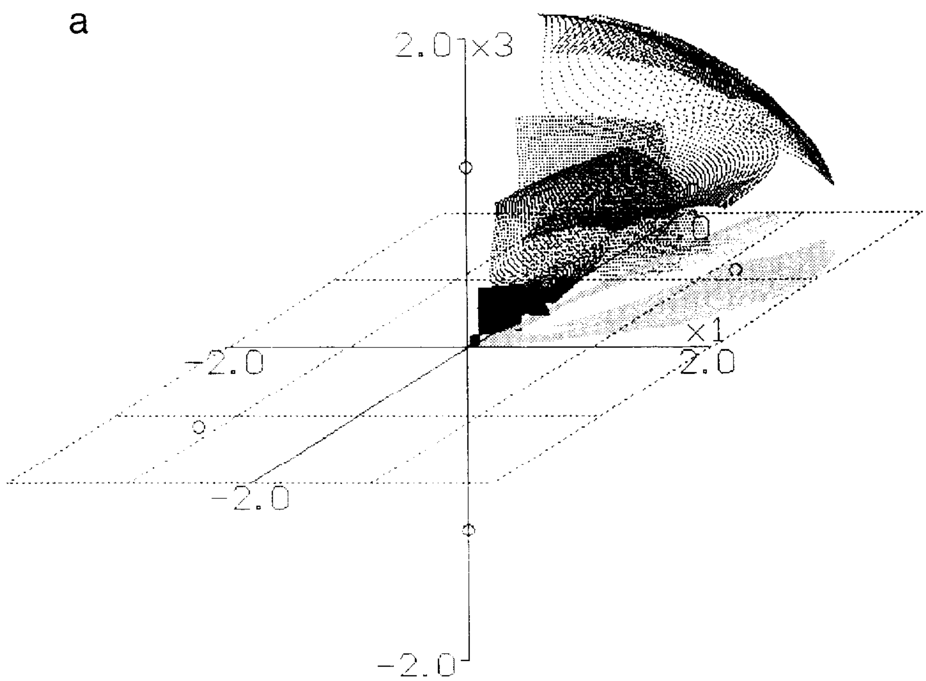
$$\boldsymbol{M}(s^*) = \boldsymbol{0}, \qquad \det D\boldsymbol{M}(s^*) \neq 0 \tag{32}$$

at some points $s^* \in \mathbb{R}^m$. Furthermore, when the second relation in (32) becomes an equality and some additional conditions on the derivatives of $\boldsymbol{M}$ hold, one obtains conditions for homoclinic tangency between these manifolds.

Applying these results to an example of a 4-D symplectic map, perturbed by conservative and dissipative terms with coefficients γ_i and δ_i ($i = 1, 2$) respectively, we first showed that when (32) holds, transversal intersections of the 2-D invariant manifolds (W^s, W^u) of a saddle point are indeed observed. As such, we interpreted the crossing of W^u points (through W^s) into the $x_n < 0$ and/or $y_n < 0$ region of the $(x_n, x_{n-1}, y_n, y_{n-1})$ space.

We have also studied the occurrence of non-transversal intersections in our example, by fixing δ_1, δ_2 and looking for γ_1, γ_2 values such that $M_1(s)$ and $M_2(s)$ satisfy tangency conditions at $\boldsymbol{s} = \boldsymbol{s}^*$ points at which they vanish. We found that, even though we could not satisfy such conditions at the *same* $\boldsymbol{s}^* = (t^*, \theta^*)$, the requirement that

$$M_i(s_i^*) = 0, \qquad \frac{\partial M_i}{\partial t}(s_i^*) = 0, \qquad i = 1, 2, \tag{33}$$



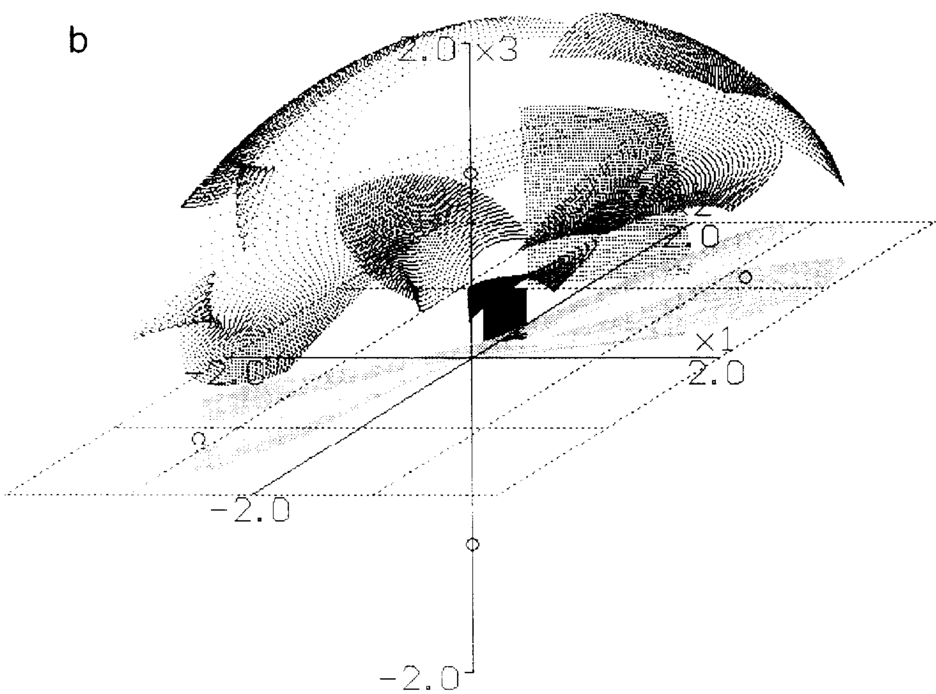
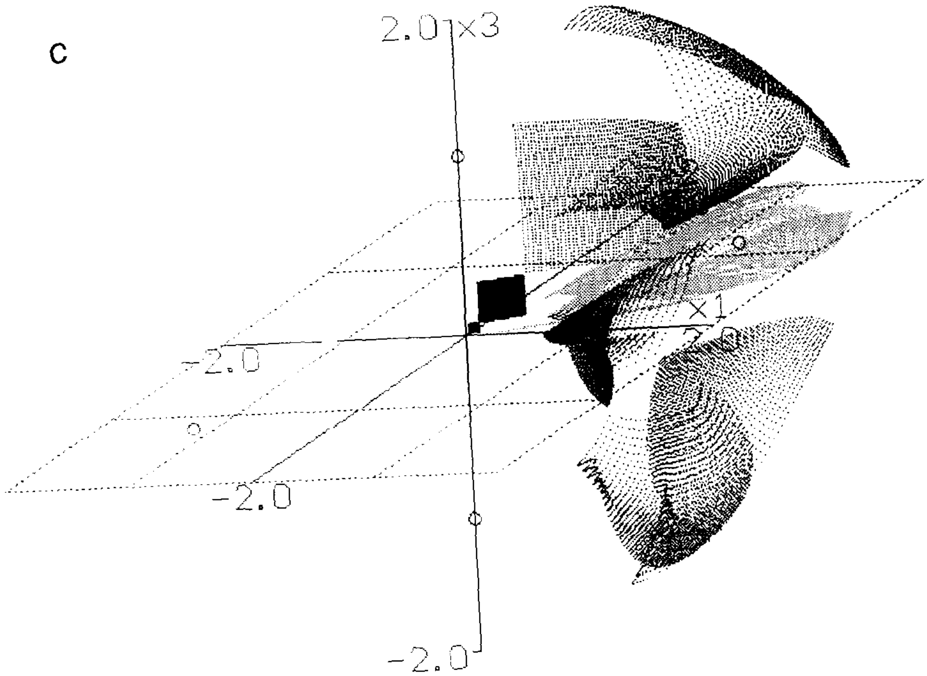


Fig. 3. Forward iterations of 400 points lying within a small parallelogram formed by the eigenvectors with positive components of the planar approximation of W^u near the origin for γ_1, γ_2 values: (a) in region I, (b) in region II, (c) in region III, (d) in region IV. Circles indicate hyperbolic fixed points. Here also $\delta_1 = 0.03$, $\delta_2 = 0.01$.

c



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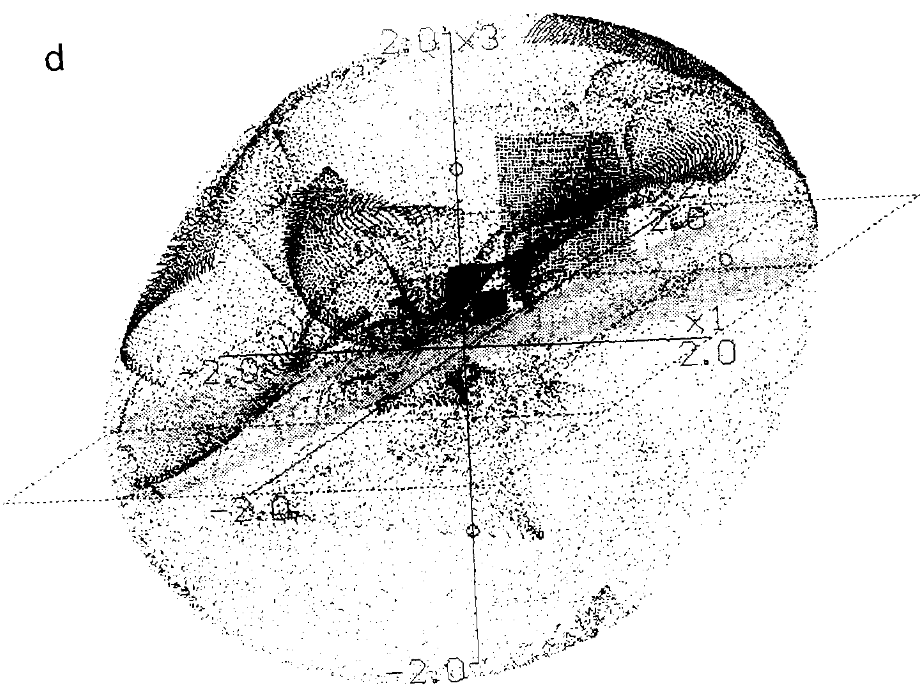

Fig. 3 (continued).

Table 1
Parameter values for non-transversal intersection at $\delta_2 = 0.01$ ($w = 1.5$)

δ_1	γ_1^{num}	γ_2^{num}	γ_1^{th}	γ_2^{th}
0.01	−0.0392	−0.0392	−0.0403	−0.0403
0.01	0.0401	0.0401	0.0403	0.0403
0.02	−0.0491	−0.0421	−0.0476	−0.0403
0.02	0.0509	0.0579	0.0329	0.0403
0.03	−0.0588	−0.0447	−0.0549	−0.0403
0.03	0.0608	0.0750	0.0256	0.0403
0.04	−0.0684	−0.0473	−0.0622	−0.0403
0.04	0.0703	0.0916	0.0183	0.0403
0.05	−0.0779	−0.0497	−0.0695	−0.0403
0.05	0.0794	0.1079	0.0110	0.0403

at $s_1^* \neq s_2^*$ and the rank condition (30), give γ_i^{th} values at which non-transversal intersections in 4-D are expected. Such intersections are indeed numerically observed, at γ_i^{num} which are in satisfactory agreement with the theoretical values, especially near the symmetric case $\delta_1 = \delta_2$.

Clearly, a lot remains to be done: Does the fact that $s_1^* \neq s_2^*$ at non-transversality imply that no homoclinic tangency in the usual sense occurs in our example? Is this a generic result or does it depend on the particular type of perturbation we have chosen?

Similarly, what about transversal intersections in regions II and III where the invariant manifolds do not extend in the full 4-D space? Is the dynamics in those regions (transient) chaotic, and how do the properties of the orbits compare with those of region IV in which homoclinic orbits seem to visit all space? These and other related questions are currently under investigation and results are expected to appear in a future publication.

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