

Simple Solution to the Nonlinear Front Problem

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(Received 17 April 1995)

The problem of determining the front speed for one-dimensional real reaction-diffusion equations is considered. A new solution to the problem, valid for a large class of functions, is proposed. In contrast with other methods, this new approach does not rely on the explicit computation of the front solutions and provides an explicit formula relating the nonlinear speed to the parameters of the equation.

PACS numbers: 82.40.Ck, 02.30.Hq, 47.10.+g

In recent years, reaction-diffusion equations have received considerable interest from both mathematicians and physicists. They have been used to model different phenomena in fluid dynamics, dendritic growth, population growth, chemical reactions, and other biological models [1–10].

We consider here the particular case of one-dimensional real reaction-diffusion equations $u_t = u_{xx} + f(u)$, where $f(u)$ has two roots at $u = 0$ and $u = u_+ > 0$ and is continuous and differentiable on $[0, u_+]$. Some solutions seem to be particularly important to describe the dynamics of such systems, the so-called *front solutions*. Front solutions are solutions of constant speed connecting the two homogeneous states $u = u_+$ to $u = 0$. They separate different regions where the system has different states and propagate perturbations from one region to another. It is observed both experimentally and numerically that, despite the fact that a system can typically accommodate families of fronts, the global nonlinear dynamics rapidly selects a unique solution. The physically relevant question is then: At what speed does the front move to the unstable state? This speed will be referred to as the *selected speed*.

Dee and Langer [11] proposed a general method for finding the selected front speed. According to their study, the preferred speed is the maximal speed of the most unstable Fourier modes. The important feature of their analysis is that the preferred front speed $c^* = 2\sqrt{\mu}$ [where $\mu = f'(0)$] is *solely* determined by the *linearized equation*. Therefore we shall refer to this preferred speed as the *linear front speed*. While this linear approach seems correct for simple systems (for instance, for all functions f concave on $[0, u_+]$), it was understood that many one-dimensional equations admit stable fronts with faster propagation speed than the linear front speed [12]. One such equation is the quintic Fisher-Kolmogorov equation [13,14]:

$$u_t = u_{xx} + \mu u + u^3 - u^5, \quad u \in \mathbb{R}. \quad (1)$$

For μ small enough, this equation has front solutions whose preferred speed \tilde{c} cannot be predicted from a linear analysis of $u_t = u_{xx} + \mu u$. The speed $\tilde{c} > c^*$ will be called the *nonlinear front speed*. These nonlinear fronts are physically relevant, since, typically, they exist for

small values of the stress parameter μ , where model equations such as (1) are valid.

Since the front solution propagates at a constant speed c and connects the state $u = u_+$ to $u = 0$, the problem of finding such a solution reduces to finding a heteroclinic solution $\hat{u}(z) = \hat{u}(x - ct)$ connecting $u = u_+$ to $u = 0$ for the ordinary differential equation obtained by the traveling wave reduction $z = x - ct$:

$$u_{zz} + cu_z + f(u) = 0. \quad (2)$$

For the one-dimensional real case, the selection principle has been completely described by Aronson and Weinberger who showed that the selected front corresponds to the slowest speed c for which positive front solutions exist [12,15]. Therefore, the question of front selection for the partial differential equation (PDE) is equivalent to finding the lowest value of c such that there exists a heteroclinic solution $\hat{u} = \hat{u}(z)$ joining the two fixed points $u = 0$ and $u = u_+$ and such that $\hat{u}(z) > 0 \forall z$. In the phase space (u, u_z) , the linear front speed corresponds to the speed for which the fixed point at the origin changes type from a sink to a node. An evaluation of the linear eigenvalues at the origin [$\lambda_{\pm} = -c/2 \pm (\sqrt{c^2 - 4\mu})/2$] leads then to the linear front speed $c^* = 2\sqrt{\mu}$. However, for a system such as (1) and for small positive values of μ , the heteroclinic connection at $c = c^*$ is such that $u(z) < 0$ in a neighborhood of the origin. Therefore one has to find $\tilde{c} > c^*$ such that the condition $u(z) > 0$ is fulfilled. It is a standard matter to show that for $c = \tilde{c}$ there exists a strong heteroclinic connection (SHC) in phase space, that is, a heteroclinic solution $\hat{u}(z)$ connecting $u = u_+$ to $u = 0$ and tangent to the strong eigendirections at the origin (corresponding to the eigenvalue λ_-).

Therefore the problem of finding the nonlinear front speed for the PDE reduces to finding the value c such that there exists a strong heteroclinic connection in phase space.

The traditional approach to computing the nonlinear front speed consists of finding the explicit solution for the front [4,7]. For instance, van Sarloos [13] proposed an ansatz for the nonlinear front solution. He was then able to find a closed form solution for the front, the analysis of which gives the nonlinear speeds. Subsequently, it was shown by Powell, Newell, and Jones [14,16] that the

nonlinear fronts could be found by applying the truncated Painlevé PDE test [17]. More generally, there has been an extensive interest in finding particular solutions of reaction-diffusion equations using different methods such as symmetry reductions [18] or direct methods [19]. However, despite the efficiency of these methods, they only apply to a restricted class of “solvable” systems, i.e., systems whose front solutions can be explicitly computed. This class of solvable systems is defined by a polynomial function $f = \mu u + \nu u^{(n+1)/2} - u^n$ [20]. More recently, new solvable cases of polynomial functions [21] or nonpolynomial functions [22] have also been found. The exact integrability methods are all based on the same idea. If the strong heteroclinic connection can be computed, then it gives the nonlinear speed. This may seem obvious, but an analytic form is not available in general. There are indeed systems for which the nonlinear fronts are proved to exist but whose closed form (henceforth, their speeds) cannot be computed [20]. The simplest example is provided by a slight modification of Eq. (1) for which $f = \mu u + u^4 - u^5$. This equation has nonlinear fronts for μ small enough; however, to date there is no available technique to find their explicit form. Can we still compute the nonlinear speeds? Is there a simple way to obtain the speeds without computing the solution? The physical problem of finding the asymptotic nonlinear front speeds for general functions remained unanswered, and it was widely believed that the nonlinear front speed cannot be computed in general. It is the purpose of this Letter to show that the nonlinear front speed can be obtained in general without computing the front solution.

Rather than considering the most general functions f , we consider a very large class of polynomial functions characterized by

$$f(u) = \mu u + g(u), \tag{3}$$

where g is a nonlinear polynomial [$g(0) = g'(0) = 0$ and is independent of μ ($\partial g/\partial \mu = 0$).

Let us now consider, in more detail, the class of the system $u_{zz} = cu_z + f(u) = 0$ with $f = \mu u + \nu u^{(n+1)/2} - u^n$, and integer n for which the exact SHC can be computed explicitly. In this case, the front solution is given by [20]

$$\hat{u} = u_+ \frac{e^{\alpha\beta z}}{(1 + e^{\alpha z})^\beta}, \tag{4}$$

with $\beta = 2/(n - 1)$ and $\alpha = (-\nu/4\beta^{3/2})(\sqrt{\beta + 1} + \sqrt{\beta + 1 + 4\mu})$.

When $\nu > 0$ and $\mu < \mu_c = 2(n + 1)/(n - 1)^2$, the nonlinear front solution is selected by the dynamics.

Let us have a closer look at the exact solutions (4). These solutions share a remarkable property: they are periodic in the imaginary time [that is, there exist $\omega \in \mathbb{R}_0$ such that $\hat{u}(z + i\omega) = \hat{u}(z)$]. Is this property of i -periodicity still valid for other systems? Although it cannot be checked directly on systems for which the exact solution is not known, a necessary condition can be easily obtained in the nonsolvable cases. Indeed, let λ_- and γ_+

be, respectively, the eigenvalues of the fixed points $u = 0$ and $u = u_+$. In the neighborhood of $u = 0$, the SHC, \hat{u} , behaves like $\hat{u} \sim Ke^{\lambda_- t}$ with i periodicity $2\pi/\lambda_-$. On the other hand, around $u = u_+$, the SHC can be written

$$\hat{u} \sim u_+ + Le^{\gamma_+ t}, \tag{5}$$

with i periodicity $2\pi/\gamma_+$.

Therefore a necessary condition for the heteroclinic solution to be periodic in the i direction is that the ratio of the two asymptotic i periods be rational:

$$\delta = -\frac{\gamma_+}{\lambda_-} \in \mathbb{Q}. \tag{6}$$

For the solutions (4), we verify that $\delta = (n - 1)/2$ is indeed rational.

By analogy with the Poincaré normal form theory, we will refer to δ as a *global resonance* (the local resonances are given by the ratio of two eigenvalues at a given point, whereas the global resonance is given by the ratio of two eigenvalues at two different fixed points).

Another striking property of the exact solutions (4) is that the ratio δ does not depend on (c, μ) but only on the degree n of the vector field. This strongly suggests that this intrinsic property will be maintained also in the nonsolvable case; that is, the global resonance δ does not depend on the parameters (c, μ) . It is plausible that the only distinction between solvable and nonsolvable cases is that in the latter case $\delta \notin \mathbb{Q}$. However, our interest here is not to characterize solvability (precise statements will be available in Ref. [23]) but rather to extract, from the particular class of solvable systems, generic properties that may hold in the general case (whether or not an explicit solution can be found). Following that idea, we conjecture that δ is, for a given $f(u)$, independent of the parameter $\mu = f'(0)$ and of the speed c . More precisely, let us now state the conjecture on which the rest of this paper is based.

Conjecture.—For a given $f(u)$ and for the values $c = \bar{c}(\mu)$ for which there exists a SHC, the global resonance δ is constant.

In other words, the global resonance δ only depends on the nonlinear structure of the vector field. This conjecture will be discussed in the last part of this paper. The independence of δ on μ can be used to find the curve $c = \bar{c}(\mu)$ for which the system admits a SH solution. Indeed, assuming that δ is known, it is a trivial matter to obtain a relation between c and μ using

$$\gamma_+(c, \mu) = -\delta\lambda_-(c, \mu), \tag{7}$$

which can be solved to obtain $c = \bar{c}(\mu)$.

However, even if δ is constant for all values of c such that $c = \bar{c}(\mu)$, we have to compute its value for at least one particular value of μ . In the problem we consider, there is one special value of μ for which the strong heteroclinic connection can be obtained exactly. Indeed, when $c = 0$ the system $u_{zz} = -f(u)$ is Hamiltonian, where $H = \frac{1}{2}u_z^2 + V(u)$ with potential $V = \int du f(u)$.

This Hamiltonian system has, for $\mu = \mu_H < 0$, a unique strong heteroclinic connection connecting $u = u_H$ to $u = 0$ defined by the condition $V(u_H) = V'(u_H) = 0$. For this value of $\mu = \mu_H$, the global resonance can be easily computed:

$$\delta = \sqrt{\frac{f'(u_H)}{\mu_H}}. \tag{8}$$

The strong heteroclinic connection for $\mu > 0$ and $c > 0$ is a continuous deformation of the subcritical ($\mu < 0$) heteroclinic connection. Relation (7) reads explicitly

$$(\delta + 1)c + \delta\sqrt{c^2 - 4\mu} - \sqrt{c^2 - 4f'(u_+)} = 0. \tag{9}$$

Therefore the speed $c = \tilde{c}(\mu)$ for which the equation admits a strong heteroclinic connection from $u = u_+$ to $u = 0$ is

$$\tilde{c} = \frac{\mu\delta^2 - f'(u_+)}{\sqrt{-\delta(1 + \delta)[f'(u_+) + \delta\mu]}}. \tag{10}$$

When μ increases, there exists a critical value of $\mu = \mu_c$, where both the linear and nonlinear fronts have the same speed. For $\mu > \mu_c$, the linear front is selected, whereas for $\mu < \mu_c$, the nonlinear front is selected. This critical value is physically interesting to know, since linear and nonlinear front speeds do not have the same dependence on the stress parameter μ . Let us stress that, in contrast to a general belief, the curves for the linear speed $c = c^*(\mu)$ and the nonlinear speed $c = \tilde{c}(\mu)$

are tangent at $\mu = \mu_c$ and not intersecting. Indeed for all values of c less than the linear speed c^* , there is no monotonic solution; therefore there is no value of μ such that \tilde{c} is less than c^* . As a consequence, the nonlinear front is always faster than the linear front except at $\mu = \mu_c$, where linear and nonlinear fronts are the same (see [24] for a complementary analysis). Equating $c^*(\mu) = 2\sqrt{\mu}$ with $c = \tilde{c}(\mu)$ given by Eq. (10), we find

$$\mu_c = -g(u_c)/u_c, \tag{11}$$

where $g(u)$ is defined in Eq. (3) and u_c is the solution of

$$g'(u_c)u_c = (\delta + 1)^2g(u_c). \tag{12}$$

As an example, let us consider the equation $u_t = u_{xx} + \mu u + \nu u^n - u^m$ with m, n integers ($m > n$) and $\nu \neq 0$. Not only does this equation admit nonlinear front solutions for μ small enough, but the explicit value of $c = \tilde{c}(\mu)$ is known only when $m = (n + 1)/2$ (the solvable case). The first step consists of determining δ . Using relation (8), and after a few algebraic computations, we find $\delta = \sqrt{(n - 1)(m - 1)/2}$. Let us note that δ does not depend on ν . Therefore, in this case, even if ν depends on the stress parameter μ the nonlinear speed can still be predicted. It can be readily verified that in general $\delta \notin 7$. The nonlinear speed is then given by relation (10), which reads

$$\tilde{c} = \frac{[\nu u_+^n(n - 1)(m + 1) - u_+^m(n + 1)(m - 1)]^2}{u_+ \delta(1 + \delta)[\nu u_+^n(\delta - n + 1) - u_+^m(\delta - m + 1)]}, \tag{13}$$

where u_+ is the positive solution of $\mu + \nu u_+^{n-1} - u_+^{m-1} = 0$.

The critical value of the stress parameter for which linear and nonlinear front speeds are equal is

$$\mu_c = \left[\frac{n - (\delta + 1)^2}{m - (\delta + 1)^2} \right]^{m-1} - \left[\frac{n - (\delta + 1)^2}{m - (\delta + 1)^2} \right]^{n-1}. \tag{14}$$

For the particular case $n = 4$ and $m = 5$, we have $\delta = \sqrt{6}$ and $\mu_c = (603 + 642\sqrt{6})/10\,000 \approx 0.2175$ (see Fig. 1).

The derivation of the nonlinear speed rests on a conjecture, namely, that for a strong heteroclinic connection, the ratio of the eigenvalues at the two fixed points depends only on the nonlinear part of the vector field. Different independent facts can be used to argue in favor of this conjecture. Most of these arguments are presented in a companion paper [23]. First, this conjecture is known to be true for the class of solvable systems mentioned in this paper. Second, the predicted nonlinear speed satisfies the known upper and lower bounds on the nonlinear speed. Third, it is possible to compute the value of δ independently at $\mu = \mu_c$. Indeed, one can use the fact that the linear and nonlinear front speeds are tangent at $\mu = \mu_c$ to compute

the value of δ . In the case where this computation can be explicitly performed, one recovers the predicted value of δ at $c = 0$ and $\mu = \mu_H$; this proves the internal consistency of the approach. Fourth, using a Melnikov-type argument, the tangent to the curve $c = \tilde{c}(\mu)$ can be found and compared at some points with the tangent predicted assuming that δ is constant along the curve. Fifth, we have tested numerically the result. We present here the case of the function $f(u) = \mu + u^4 - u^5$, first introduced as

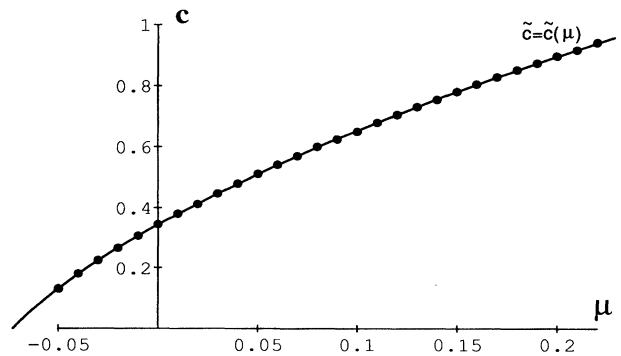


FIG. 1. The numeric (dots) and analytic curve for the nonlinear front speed of the system $u_t = u_{xx} + \mu u + u^4 - u^5$

a counterexample in Ref. [20]. The numerical procedure consists of integrating Eq. (2) in phase space (u, u_z) along the stable manifold (forward) to $u = 0$ and unstable manifold (backward) to $u = u_+$. At a given point between $u = 0$ and $u = u_+$ and for every value μ , we can tune c such that the manifolds intersect in order to find the value \tilde{c}_{num} of intersection. The results are represented by dots on Fig. 1.

The PDE dynamics selects the nonlinear front whenever the speed is faster and the front steeper than the linear front. This selection principle can be understood in the phase space of the reduced ordinary differential equation. It is equivalent to the existence of a strong heteroclinic curve joining the unstable fixed point to the stable one. In turn, this special heteroclinic structure is related to special properties in the complex plane. Namely, in the solvable cases, the heteroclinic solution is periodic in the imaginary direction, while it is only quasiperiodic in the nonsolvable cases. Therefore, using this periodicity (or quasiperiodicity), the nonlinear front speed can be obtained by assuming that this property is not affected by the linear portion of the vector field around the asymptotic state. By contrast, the linear front speed does not depend on the nonlinear part of the vector field. The method of global resonances uses the linearized (local) eigenvalues of the two fixed points together to compute the periodicity, or lack thereof, of the heteroclinic solution in the imaginary direction. It was used efficiently in the computation of the nonlinear front speeds. In Ref. [23], other examples are treated along the same lines. In particular, we show how to generalize the results to general vector fields whose nonlinear part also depends on the stress parameter. This study also addresses an interesting problem in dynamical systems theory which to the best of my knowledge has not been solved satisfactorily: Given an n -dimensional system with homoclinic (heteroclinic) connections, find the curve in parameter space where special connections (such as the SHC) exist. I believe that the method of global resonances might be a first attempt to answer this question.

Finally, let us stress the importance of the one-dimensional problem. The nonlinear front speeds for higher-order equations, such as the Swift-Hohenberg equation, is obtained through perturbation techniques around the one-dimensional problem [13,25]. Therefore a complete knowledge of the one-dimensional problem is necessary to obtain information concerning the nonlinear fronts of other physically relevant equations.

The author thanks M. Tabor for introducing him to the problem and J. Powell for many fruitful discussions.

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