Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations

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The integrability of systems of ordinary differential equations with polynomial vector fields is investigated by using the singularity analysis methods. Three types of results are obtained. First, a general relationship between the degrees of first integrals and the so-called Kowalevskaya exponents is derived. Second, it is shown that all solutions of algebraically integrable systems can be expanded in Puiseux series. Third, a new method to study partially integrable systems is studied. These different aspects allow us to study algorithmically the integrability, partial integrability, and nonintegrability of differential systems. © *1996 American Institute of Physics.* [S0022-2488(96)01103-3]

I. INTRODUCTION

In recent years many works have focused on the definition of integrability for dynamical systems. However, it is well known that ''integrability'' in a general sense is ill defined. This is mainly due to a confusion between Hamilton theory, dynamical systems approach, and singularity analysis. Each field has a different definition for ''integrability'' which makes sense within the theory. The difficulty arises when one tries to establish possible relationships between different fields. The main problem treated here is to find a connection, if any, between the Painlevé property and another notion of integrability which could be used to effectively build the solutions or gain some global knowledge on the dynamics in phase space. While it is widely believed that the Painlevé property is incompatible with chaotic motions, there is to date no rigorous proof of this simple statement. From the other point of view, it is known that Liouville integrability is not directly related to the Painlevé property. Therefore, if we want to draw arrows between these different fields it is necessary to introduce some rigorous notions of integrability which could be explicitly related to the Painlevé property, or, at least, to the Painlevé test. More generally, one needs a simple test for the existence or nonexistence of first integrals in a given function space (polynomial, rational, algebraic,...) and it is the purpose of this paper to show that singularity analysis provides it.

The algebraic integrability for an n-dimensional systems of ODEs with rational vector fields is defined as the existence of $(n-1)$ algebraic first integrals. This notion of integrability is very strong. Indeed, Liouville integrability for an n degrees of freedoms Hamiltonian only requires the existence of n single-valued first integrals; the remaining $(n-1)$ angle variables expressed as closed one-form are not, in general, algebraic or even single-valued first integrals. Liouville integrability is therefore a much weaker statement on the singularities than algebraic integrability. This explains why it has not been possible so far to decide from the singularity analysis the Liouville integrability for general Hamiltonians. From the other point of view, the notion of algebraic integrability constrains the solutions of the systems in such a way that general statements on the meromorphicity of the solutions are possible.

Another problem relates to partial integrability, that is the existence of a certain number of

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first integrals less than the number required for the complete integration. The singularity analysis has been a successful tool for finding integrable systems. Many new systems, Hamiltonian and non-Hamiltonian, have been built.[1] However, most of the Hamiltonian systems although Liouville integrable cannot be detected by the singularity analysis. In the same way, most of the dynamical systems admit a few invariants related to physical conservation laws, but only a handful of them will admit enough constants of motion to effectively build the complete analytical global solution. More generally, there is no decision procedure to test the simple question: Is there a polynomial first integral for a given systems of ODEs? Or, alternatively: Is there a bound for the degree of polynomial first integrals?

The explicit computation of first integrals is not an easy task. The different techniques depend on the space of functions considered for the first integrals. If we are interested in polynomial or rational first integral, then all methods of explicit construction rely on the same idea. Consider a system $S:\dot{x}=f(x)$ where f is a given polynomial vector field and let us look for a first integral of *given* degree d. We insert a polynomial or rational ansatz in the system and look for the coefficients. If no first integral is found, then one has to increase the value of d and, with good luck and prayer, hope to obtain a first integral.[2–5] Some of these methods are based on the Carleman procedure for finding first integrals.[6–10]

The main problem of all these methods is to set the degree d. We show here that the degree of a first integral is related to the Kowalevskaya exponents. This relation gives a first choice for the possible values of the degree d.

From the other point of view, it is sometimes possible to prove nonintegrability, that is, the nonexistence of constants of motion. For Hamiltonian systems, there is the Ziglin theory for the n degrees of freedom systems. This theory has been proved to be useful for the following systems: the motion of rigid body around a fixed point,[11,12] homogeneous potentials,[13,14] the Toda lattices,[15,16] a perturbed Kepler potential,[17] nonhomogeneous potentials,[18] and a reduced Yang–Mills potential.[19]

Ziglin's theory is based on the monodromy properties around particular solutions (straight line notions). However, it is quite difficult to apply in general and is limited to low degrees of freedom and, mostly, homogeneous potentials.

For non-Hamiltonian systems, only a few results on nonintegrability are available. Let us mention a recent work for homogeneous vector fields due to Moulin-Ollagnier *et al.*,[20] where some new results on the nonexistence of polynomial first integrals of motion based on algebraic considerations are given.

Both theories, integrability and nonintegrability, are dichotomous in the sense that, if integrability is not proved by the singularity analysis, then nothing can be said about the existence of first integrals. Nonintegrability, in the same way, is a statement on the nonexistence of at least one first integral.

However, most of the systems encountered in physics do not fall in the set of completely integrable or completely nonintegrable systems. Indeed, if a system admits one or two first integrals, then nonintegrability cannot be proved in general. In Sec. V, we show that singularity analysis can be effectively used to find partially integrable cases. Therefore, it can be applied to n degrees of freedom Hamiltonian systems where only $(n-1)$ first integrals have to be built to complete the Liouville integrability. These necessary conditions provide a direct and algorithmic proof of the nonintegrability of these systems.

This paper is divided as follows. In Sec. II, I review the definitions and the relevant works concerning algebraic integrability for systems of ODEs. In Sec. III, I present a new and general result relating the degrees of first integrals for homogeneous vector fields with the Kowalevskaya exponents. In Sec. IV, the main theorems of this paper are presented. It is proved that algebraic integrability implies that the solutions can be expanded in Puiseux series. Finally, in Sec. V, I present a new method for finding necessary conditions for partial integrability for nonhomogeneous vector fields whose homogeneous part is partially integrable.

II. ALGEBRAIC INTEGRABILITY

I briefly recall the main general results concerning the existence of first integrals.

A. Definitions and preparations

Consider a system of n first-order ODEs:

$$S(f;x,t):\dot{x}=f(x), \tag{2.1}$$

where $x\in\mathbb{R}^n$ and $f_i(x)$ are polynomial functions of x.

A *first integral* $I=I(x,t)$ of $S(f;x,t)$ is a nonconstant function of (x,t) with the property

$$\nabla I.f+\frac{\partial I}{\partial t}=0. \tag{2.2}$$

This relation holds if and only if $I(x,t)$ is constant along all particular solutions $x=x(t)$ of S.

Definition 2.1: An algebraic function $I(x)=C$ is a solution of

$$q_0+q_1C+q_2C^2+\cdots+q_{s-1}C^{s-1}+C^s=0, \tag{2.3}$$

where $q_i(x)$ are rational functions of x, and s is the smallest positive integer for which such a relation holds. The relation (2.3) is referred to as the minimal polynomial of I.

Let us recall that two first integrals are *independent* if there is at least one point $x_0\in\mathbb{R}^n$ such that their gradients are linearly independent. In the same way, l first integrals are *independent* if there is a point $x_0\in\mathbb{R}^n$ such that

$$\operatorname{rank}(\nabla I_1(x_0),\nabla I_2(x_0),\ldots,\nabla I_l(x_0))=l. \tag{2.4}$$

Let us note that if $x_0=x(t_0)$ is the initial condition, the first integrals remain independent for all $x(t)$ $(t\in\mathbb{R})$ solutions of S with initial conditions x_0. Indeed, the gradients of the first integrals are solutions of a system of linear differential equations, the *adjoint variational equations*. This results from a general property of linear differential equations that linearly independent solutions remain independent under the flow.[21]

Two different notions of algebraic integrability were alternatively used in the literature. The weaker definition is an extension of the Hamilton–Jacobi theorem to more general vector fields:

Definition 2.2: The system S is algebraically integrable in the weak sense if there exist k independent algebraic first integrals $I_i(x)=K_i$ $(i=1,\ldots,k)$. These k first integrals define an $(n-k)$-dimensional algebraic variety. In addition, there must exist other $(n-1-k)$ independent first integrals given by the integral of a total differential defined on the algebraic variety:

$$J_i=\sum_{j=1}^{n-k}\int^{x_j}\phi_{ij}(x)dx_j, \quad i=1,\ldots,n-1-k, \tag{2.5}$$

where $\phi_{ij}(x)$ are algebraic functions of x.

This definition seems to be useless for non-Hamiltonian systems where the existence of a total differential is not *a priori* known. However, in some cases such a situation may occur.[22] A classical example is given by the six-dimensional Euler equations for the rigid body motion around a fixed point where only four first integrals are required to complete the integration, the fifth one being given by the integral of a total differential (Ref. 23, p. 108). However, it is understood that this definition is, in most cases, likely to be applicable to Hamiltonian systems.

The stronger definition of algebraic integrability is equivalent to the weak definition with the condition $k=n-1$.

Definition 2.3: The system S is algebraically integrable if there exist $(n-1)$ *independent algebraic first integrals* I_i $(i=1,\ldots,n)$.

Still, there exist other definitions of algebraic integrability for Hamiltonian systems: the so-called *algebraic complete integrability* introduced by Adler and van Moerbeke[24] and the *hyper-elliptically separable systems* by Ercolani and Siggia.[25] Their definitions cover systems which can be integrated in terms of Abelian functions. Whereas their approaches are mainly geometric, our approach is algebraic in the sense that we are mainly interested in showing the existence of algebraic functions as constants of the motion rather than proving some particular structure for the complexified phase space.

B. Reduction of algebraic first integrals

The algebraic first integrals can be reduced to rational first integrals. Indeed, we prove that any algebraic integral is algebraically compounded from weight-homogeneous rational first integrals. This can be summarized by the following result:

Lemma 2.4: If the system $S:\dot{x}=f(x)$ *has* $l(1\leqslant l\leqslant n-1)$ *independent algebraic first integrals, then there exist* l *independent rational first integrals.*

We now give a short proof of this result:

Proof 2.4: Let $I(x)=C$ be a nontrivial algebraic first integral and $P(C)=q_0+q_1C+\cdots+q_{s-1}C^{s-1}+C^s$ its minimal polynomial. We apply the operator $f.\boldsymbol{\nabla}$ on the polynomial:

$$\begin{aligned} f.\boldsymbol{\nabla}(P(C))=0=(f.\boldsymbol{\nabla}q_0+Cf.\boldsymbol{\nabla}q_1+\cdots+C^{s-1}f.\boldsymbol{\nabla}q_{s-1}) \\ +(f.\boldsymbol{\nabla}C)(q_1+\cdots+(s-1)q_{s-1}C^{s-2}+sC^{s-1}). \end{aligned} \tag{2.6}$$

Using the fact that $I(x)=C$ is a first integral ($f.\boldsymbol{\nabla}C=0$), one finds

$$(f.\boldsymbol{\nabla}q_0+Cf.\boldsymbol{\nabla}q_1+\cdots+C^{s-1}f.\boldsymbol{\nabla}q_{s-1})=0. \tag{2.7}$$

This is a polynomial of degree smaller than the minimal polynomial $P(C)=0$, so that we have $f.\boldsymbol{\nabla}q_i=0$, $i=1,\ldots,s-1$. The first integral $C=I(x)$ is nontrivial (different from a constant), so that at least one of the $q_i(x)$ is nontrivial [if all q_i are trivial, so is $I(x)$]. This q_i is a nontrivial rational first integral of the system S.

Now, consider another independent algebraic first integral $I'(x)=C'$. Its minimum polynomial reads $P(C')=q_0'+q_1'C+\cdots+q_{s'-1}'C^{s'-1}+C^{s'}$. We know from the previous paragraph that each q_i' is a first integral. It follows from the independence of I with I' that there exist (i,j) $(i<s-1,j<s'-1)$ so that the two rational functions q_i and q_j' are nontrivial and independent (if all rational functions q_i and q_i' are dependent, so are the first integrals). Therefore, these two rational functions are independent nontrivial rational first integrals. Proceeding so, we can build l independent rational first integrals from the l algebraic first integrals. □

Let us note that this result was already contained in a paper by Bruns (1887), in a different setting (see Ref. 26).

We conclude that the study of algebraic first integrals reduces to the study of rational first integrals.

C. Elementary first integrals

Is there a special place for algebraic first integrals among other types of first integrals? Why not study the existence of more complex or peculiar forms of first integrals? The answers to these questions lie in a result of Prelle and Singer[27] (see also Refs. 28 and 29). They show that if there exists an *elementary first integral* (that is, a first integral built up from rational functions using exponentiation, integration, and algebraic functions) for a system of first-order ODEs with polynomial vector fields, then it is of the form

$$I_0(x)+\sum c_i \log I_i(x), \tag{2.8}$$

where $I_i(x)$ are algebraic functions.

Therefore, algebraic functions play a special role in the class of elementary functions since all elementary first integrals can be built out of them. Our interest is to find the relationship between the existence of first integrals and the local single-valuedness of the solution. It has been shown by Ishii[30] that the appearance of a logarithmic dependence in the first integrals implies, in general, that the solutions are multivalued in a neighborhood of their singularities. The solutions lie then on a Riemann surface. The consequence is that the system has no Puiseux expansion with $(n-1)$ arbitrary coefficients around the movable singularities. In other words, infinitely many-valuedness of a first integral brings infinitely many-valuedness of the solutions. We will therefore focus on the existence of algebraic first integrals as the first building blocks of elementary first integrals. The next step is to show that the single-valuedness of first integrals implies the single-valuedness of the solutions.

D. Scale invariant systems

The construction of a first integral relies on a decomposition of the vector field S in homogeneous and nonhomogeneous components. In order to build explicitly a first integral we start from a truncation of the vector field retaining the higher nonlinear terms and then we consider lower-order corrections. As a consequence, we first focus our study on a particular class of systems S which exhibits some particular scaling properties:

Definition 2.5: The system S:$\dot{x}=f(x)$ is similarity invariant if there exists $g \in \mathbb{Q}^n$ such that S is invariant under the transformation

$$x \to \alpha^g x, \quad t \to \alpha^{-1} t \tag{2.9}$$

for all $\alpha \in \mathbb{R}_0$.

We shall refer to g as the *weight* of S. More generally, a function $F(x,t)$ is *weight-homogeneous* with respect to g of *weighted degree d* if

$$F(\alpha^{-1}t,\alpha^g x)=\alpha^d F(t,x). \tag{2.10}$$

As a consequence, a system $S:\dot{x}=f(x)$ is similarity invariant if each component f_i of the vector field is weight-homogeneous of weight g_i+1 with respect to the weight g. Weight-homogeneous functions are the natural generalization of homogeneous functions and most of the properties of homogeneous functions can be readily translated in terms of weights.

III. ALGEBRAIC INTEGRABILITY FOR HOMOGENEOUS SYSTEMS

A. Yoshida's analysis

One of the pioneering works in the domain is due to Yoshida.[31,32] Using a singularity-analysis-type method, he was able to derive necessary conditions for algebraic integrability.

Consider a similarity-invariant system (w.r.t. a weight g) $S:\dot{x}=f(x)$. The interest of similarity-invariant systems lies in the existence of particular scale-invariant solutions of the form

$$x=ct^{-g}, \tag{3.1}$$

where the coefficients $c \in \mathbb{C}^n$ are given by the algebraic equation

$$f(c)+cg=0. \tag{3.2}$$

For a given g, there may exist different sets of values c which will be referred to as different *balances*. We now consider one of these solutions and we introduce the matrix K:

$$K = Df(c) + \mathrm{diag}(g), \tag{3.3}$$

where $(Df(c))_{ij} = (\partial f_i/\partial x_j)(c)$ is the Jacobian evaluated on $x = c$.

Here again, the eigensystem of matrix K can be used to build particular solutions to the variational equations. Let the *Kowalevskaya exponents* be the eigenvalues of K. (Sophia Kowalevskaya was the first to introduce the determinant of K to compute the Laurent series solutions of the rigid body motion: ''Afin que les séries... contiennent le nombre suffisant de constantes arbitraires, il faut que le déterminant de ces équations linéaires ...s'évanouisse pour cinq valeurs différentes de m égales à des nombres entiers positifs.''[33])

It can be shown that there always exists a Kowalevskaya exponent $\rho = -1$ related to the arbitrariness of t_0.

Yoshida's results are twofold. First, he proves that, under certain conditions, the weighted degree of a first integral is a Kowalevskaya exponent. Second, he shows that if one of the Kowalevskaya exponents is not rational, then the system cannot be algebraically integrable.

Theorem 3.1 (Yoshida): *Let $I(x)$ be a weight-homogeneous first integral of weighted degree d for the similarity invariant system S. Assume that $\nabla I(c)$ is not identically zero for at least one choice of c. Then, d is a Kowalevskaya exponent.*

1. Example: A Hamiltonian system

As an example, we study the following three degrees of freedom Hamiltonian system:[34]

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (x_1^4 + 16x_2^4 + \mu x_3^4 + 12x_1^2x_2^2). \tag{3.4}$$

This Hamiltonian system is weight-homogeneous with respect to the weights $g = (1,1,1,2,2,2)$ [where $(x_1,x_2,x_3,x_4,x_5,x_6) = (x_1,x_2,x_3,p_1,p_2,p_3)$]. Moreover, it is integrable with second and third constants of motion given by

$$C_1 = p_3^2 + 2\mu x_3^4, \tag{3.5}$$

$$C_2 = x_2p_1^2 - x_1p_1p_2 - 8x_1^2x_2^3 - 4x_1^3x_2^2. \tag{3.6}$$

The weights of H, C_1, C_2 w.r.t. g are, respectively, $d_h=4$, $d_1=4$, $d_2=5$. The first step of Yoshida's analysis consists of finding all the possible dominant balances, that is, the scale-invariant solutions $x_i = c_i t^{-g_i}$. We found 24 different solutions for c_i. For each dominant balance, we can compute the Kowalevskaya exponents ρ using relation (3.3) and the gradients of the first integrals estimated on the scale invariant solutions $x = ct^{-g}$. As an example, we give three different dominant balances in order to illustrate Yoshida's theorem:

$$c = \left(\frac{i}{\sqrt{2}}, 0, \frac{i}{\sqrt{2\mu}}, \frac{-i}{\sqrt{2}}, 0, \frac{-i}{\sqrt{2\mu}}\right), \quad \rho = \{-2,-1,-1,4,4,5\},$$

$$\nabla H(c) \neq (0), \quad \nabla C_1(c) \neq (0), \quad \nabla C_2(c) \neq (0);$$

$$\begin{gathered} c = \left(\frac{1}{2}, \frac{i}{2\sqrt{2}}, \frac{i}{\sqrt{2\mu}}, -\frac{1}{2}, \frac{-i}{2\sqrt{2}}, \frac{-i}{\sqrt{2\mu}}\right), \quad \rho = \{-2,-1,-1,2,4,5\}, \\ \nabla H(c) \neq (0), \quad \nabla C_1(c) = (0), \quad \nabla C_2(c) \neq (0); \end{gathered} \tag{3.7}$$

$$c=\left(-\frac{1}{2},\frac{-i}{2\sqrt{2}},0,-\frac{1}{2},\frac{i}{2\sqrt{2}},0\right),\quad \rho=\{-2,-1,1,2,4,5\},$$

$$\nabla H(c)\neq(0),\quad \nabla C_1(c)=(0),\quad \nabla C_2(c)=(0).$$

In the first case, the gradients of the first integrals do not vanish on the scale-invariant solution. As a consequence, d_h, d_1, and d_2 are Kowalevskaya's exponents. In the second case, ∇C_1 vanishes identically and d_1 is not a Kowalevskaya exponent, while in the third case only $\nabla H(c)$ does not vanish.

Although this result was the first bridge between the degrees of first integrals and the singularity analysis, it is not of great predictive power. Indeed, while the Kowalevskaya exponents can be computed in a finite procedure, the functional form of the first integral is not known *a priori*. Therefore, the first integrals may not satisfy the assumptions. In particular, it does not forbid the existence of a first integral of higher degree for which $\nabla I(c)$ could vanish identically. We will come back to this problem in the following sections.

Let us note that the converse statement holds. Indeed, it will be proved in the next section that $\nabla I(c)\neq(0)$ if and only if d (the degree of I w.r.t. to g) is a Kowalevskaya exponent for the balance under consideration (where d is considered here with the proper algebraic multiplicity).

Another interesting point is that this result seems to be valid outside the class of algebraic first integrals. Indeed, Yoshida's argument does not rely on the fact that I is an algebraic function but only on the weight-homogeneity of the vector field and the first integral.

For Hamiltonian systems there is an interesting relation between the Kowalevskaya exponent which was first pointed out by Yoshida and given in its final form by Lochak:[35]

Proposition 3.2 (Lochak, 1985): Let S be a system whose Hamiltonian is H. If ρ is a Kowalevskaya exponent for the system S, then so is $h-1-\rho$ (where h is the weighted degree of the Hamiltonian H).

In other words, the Kowalevskaya exponents always come by pairs for Hamiltonian systems. This is analogous to the linearized eigenvalues at a fixed point (this analogy is more than formal and can be made rigorous).

The next statement is the main result of Yoshida, it connects the occurrence of irrational Kowalevskaya exponents with nonintegrability:

If the system S is algebraically integrable in the weak sense, then all Kowalevskaya exponents are rational.

However, despite the fact that this result has been widely applied and frequently verified, this last statement is not correct as illustrated in the next example due to Kummer *et al.*:[36]

$$H=p_1(p_1^2+x_1^2)+\omega x_1(p_2^2+x_2^2). \tag{3.8}$$

It can be easily verified that this system has Kowalevskaya's exponents: $\{-1,3,1\pm 2i\omega\}$ for the similarity invariant solution $(x_1,x_2,p_1,p_2)=(-1,0,0,0)t^{-1}$. However, there exists a second polynomial first integral:

$$I=p_2^2+x_2^2. \tag{3.9}$$

The correct statement of Yoshida's theorem is only related to the stronger definition of algebraic integrability:

Theorem 3.3: *If the system S is algebraically integrable, then all Kowalevskaya exponents are rational.*

To the best of my knowledge, there is not published proof of this basic result. Therefore, a proof will be given in Sec. III D as a corollary of a more general result (see Proposition 3.5).

Although Yoshida's statement was not correct in full generality, Yoshida managed to prove it in a particular case using Ziglin's theory of nonintegrability.[11,12] He studied the case of n degrees of freedom Hamiltonian systems with diagonal kinetic contribution and homogeneous potential:

$$H=\tfrac{1}{2}(p_1^2+\cdots+p_n^2)+V(q_1,\ldots,q_n), \tag{3.10}$$

where $V(x)$ is homogeneous of degree k but $k\neq 0,\pm 2$. The Kowalevskaya exponents always come by pairs $\rho_i+\rho_{i+n}=(k+2)/(k-2)$, so that we can define the difference between two exponents of each pair $\Delta\rho_i=\rho_{i+n}-\rho_i$.

Theorem 3.4 (Ref. 14): *If the n numbers $\Delta\rho_i$ are $\mathbb{Q}$-independent, then the Hamiltonian system has no additional first integral beside the Hamiltonian itself.*

As a corollary, for the case of planar Hamiltonian systems with homogeneous potential, we obtain the following: If the Hamiltonian $H=\frac{1}{2}(p_1^2+p_2^2)+V(x_1,x_2)$ possesses a second invariant, then the Kowalevskaya exponents are rational.[14]

B. The generalized Kowalevskaya exponents

Consider a system $S(f;x,t)$, and assume for the time being that f is weight-homogeneous. Beside the Kowalevskaya exponent there is yet another set of indices that can be defined, the so-called *resonances* of the Painlevé test.[1] The Painlevé test is an algorithmic procedure which provides necessary conditions for the Painlevé property. Essentially, the Painlevé test checks the formal existence of Laurent series as a solution. Let us recall the main ingredients of the Painlevé test:

The first step consists in finding all the truncations of $\hat{f}$ of the vector field

$$\dot{x}=f(x)=\hat{f}(x)+\check{f}(x) \tag{3.11}$$

such that the *leading behavior* $x=\alpha(t-t_*)^p$, $\alpha\in\mathbb{C}_0^n$, is an exact scale-invariant solution of the homogeneous system

$$\dot{x}=\hat{f}(x), \tag{3.12}$$

where $p\in\mathbb{Q}^n$ with at least one negative component.

It is also required that $\check{f}(x)=\Sigma_i f^{(i)}(x)$ is *not dominant*, that is, at the singularity,

$$\check{f}^{(i)}(x)(\alpha(t-t_*)^p)=\gamma^{(i)}(t-t_*)^{p+q^{(i)}-1} \tag{3.13}$$

with $q^{(i)}\in\mathbb{N}_0^n$.

Each *balance* (α,p) defines a different expansion.

The second step is the computation of the *resonances*. Each balance defines a new set of resonances. These resonances are the indices j of the coefficients a_j in the Laurent series at which arbitrary constants first appear. It is a standard matter to show that these resonances are given by the eigenvalues of the matrix R:

$$R=D\hat{f}(\alpha)-\mathrm{diag}(p) \tag{3.14}$$

where again $D\hat{f}(\alpha)$ is the Jacobian matrix evaluated in α.

The resonances are labeled r_i, $i=1,\ldots,n$, with $r_1=-1$. The necessary condition for the existence of the Laurent series, in this set of variables, is that all resonances are integer $(r_i\in\mathbb{Z})$.

The third and last step of the Painlevé test consists of checking that the arbitrariness of the coefficient a_r for the full system (3.11) does not introduce incompatible constraints on the coefficients $a_j(j<r)$. This is achieved by computing all coefficients in the Laurent expansions up to the highest resonances.

We have introduced two set of indices for a given vector field f: the resonances of the Painlevé test and the Kowalevskaya exponents. The main difference is not in the definition of the exponents (they are both obtained as the eigenvalues of a matrix built on the Jacobian matrix of a particular solution), but rather on the choice of the particular solution. Indeed, in the Painlevé test, one looks for a particular solution of $S(f^{(0)})$ under the form $x=\alpha(t-t_*)^p$ where $\alpha\in\mathbb{C}_0^n$ while in Yoshida's analysis the particular solution of $S(f)$ is $x=c(t-t_*)^{-g}$ with $c\in\mathbb{C}^n$ (not in $\mathbb{C}_0^n$!). This rather subtle difference introduces a shift in the exponents that we now make explicit. Let us stress before proceeding that both sets of exponents have their own interest and they correspond to different types of analysis. Yoshida's analysis stresses that the weight of polynomial functions and the existence of series involving logarithmic terms is not a relevant feature since only the rationality of the Kowalevskaya exponents comes into play. From the other point of view, the Painlevé test is designed to test the existence of Laurent series as formal solution. It is common to mistake both sets. This is why we explained in length these differences.

Consider the system $S(f;x,t)$ (where f is weight-homogeneous of weight $g\in\mathbb{Q}^n$). We build a particular solution of this system using the singularity analysis, that is, a balance (α,p) of order l. Therefore, there is a truncation of the vector field $f=f^{(0)}+f^{(1)}+\cdots+f^{(l')}$ $(1\leqslant l'\leqslant n-l)$ and a particular solution of $S(f^{(0)})$:

$$x=\alpha(t-t_*)^p, \tag{3.15}$$

where, without loss of generality (i.e., up to a permutation of indices), we have α_i arbitrary for $i=l+1,\ldots,n$.

For each balance (α,p), there corresponds a particular solution of $S(f)$ as defined in the previous section:

$$x=c(t-t_*)^{-g} \tag{3.16}$$

where $c_i=\alpha_i$, $i=1,\ldots,l$, and $c_i=0$, $i=l+1,\ldots,n$.

Therefore, to a balance of order l, there corresponds a particular solution with $n-l$ vanishing entries. Moreover, from the homogeneity of the vector field, we can deduce the relation $f(c)=f^{(0)}(\alpha)$ and the shift between the dominant exponents p and the weights g. Let n_i be the number of nonvanishing components of $f^{(i)}$, then we have

$$p_j=-g_j+q^{(i)}, \quad j=n_{i-1}+1,\ldots,n_{i-1}+n_i, \quad i=0,\ldots,l', \tag{3.17}$$

with $n_{-1}=0$ and $q_0=0$

In the same way it is possible to build the eigenvalues of the Kowalevskaya matrix (3.3) from the resonances:

$$\rho_i=r_i \quad \text{for all } i \text{ such that } r_i\neq 0, \tag{3.18}$$

$$\rho_i=q_i \quad \text{with multiplicity } n_i. \tag{3.19}$$

Now, the correspondence (3.18) can be used as a new definition for the Kowalevskaya exponents for nonhomogeneous vector fields $S(f;x,t)$. Let us note that the number of these *generalized Kowalevskaya exponents* can exceed the number of variables n. Only n of these m exponents corresponds to independent arbitrary constants. However, all exponents may be used to test the existence of homogeneous first integrals.

C. Necessary conditions for algebraic integrability

Yoshida's result only concerns complete integrability. However, following the same arguments, his result can be generalized to study partial or nonintegrability. In this section I establish a fundamental equality between the Kowalevskaya exponents for nonhomogeneous systems and the existence of first integrals.

Proposition 3.5: If there is l independent algebraic first integrals $I_1,...,I_l$ of weighted degrees $d_1,\ldots,d_l$ for a system $\dot{x}=f(x)$, then there is l independent linear relations:

$$\sum_{j=1} N_{ij}\rho_j=d_i, \quad i=1,\ldots,l, \tag{3.20}$$

with $N_{ij}\in\mathbb{Z}$.

Proof 3.5: According to Lemma 2.4, we consider l independent rational weight-homogeneous first integrals. The proof is divided into two parts. First, we show that the existence of a commensurate relation between the degrees of the first integrals and the Kowalevskaya exponents. Second, we show that these relations are linearly independent as a consequence of the independence of the l first integrals.

The similarity invariant solution $x=ct^{-g}$ is the first term of a formal solution around a singularity t_*:

$$x=(t-t_*)^{-g}\sum_{i_j}^{\infty} a_{i_1\ldots i_m}\prod_{j=1}^{m} \xi_j^{i_j}, \tag{3.21}$$

where $\xi_j=\lambda_j(t-t_*)^{\rho_j}$, the λ_js are independent arbitrary constants, and $\rho_1,...,\rho_m$ are the positive Kowalevskaya exponents. The coefficients $a_{i_1\ldots i_m}$ are polynomial in $\log(t-t_*)$ and the sum Σ_{i_j} is taken over all positive i_j.

Now, consider the weight-homogeneous rational first integral I:

$$I=\frac{\Sigma c_i x^{E_i}}{\Sigma d_j x^{F_j}} \tag{3.22}$$

with $(E_i-F_j).g=d\forall i, j$.

This first integral I can be evaluated on $x=x(t-t_*)$ as a function of $(t-t_*)$ by inserting (3.21) in (3.22):

$$I=(t-t_*)^{-d}\sum_{i_j} K_{i_1\ldots i_m}\prod_{j=1}^{m} \xi_j^{i_j}, \tag{3.23}$$

where $i_j\in\mathbb{Z}\forall j$.

On the lhs of (3.23), I is an arbitrary constant, therefore, there exists on the rhs of (3.23), a combination of the arbitrary constants λ_i to order $(t-t_*)^0$, that is, there exists at least one set of integers $\{i_1,\ldots,i_j\}$ such that $K_{i_1,\ldots,i_m}\neq 0$ and $i_1\rho_1+\cdots+i_m\rho_m=d$ with $i_j\in\mathbb{Z}\forall j$.

We digress at this point to notice that, had we considered a *polynomial* first integral, rather than a *rational* first integral, we would have obtained the relation $i_1\rho_1+\cdots+i_m\rho_m=d$ with $i_j\in\mathbb{N}\forall j$.

We now have to prove the independence of the linear relations. To do so, we consider two first integrals I and I' and show that at least two independent linear relation $i_1\rho_1+\cdots+i_m\rho_m=d$ can be obtained. The result for the independence of l linear relations naturally follows.

Let us introduce the variables $\{\xi_1=\lambda_1(t-t_*)^{\rho_1},\ldots,\xi_m=\lambda_m(t-t_*)^{\rho_m},\xi_{m+1}=\lambda_{m+1}(t-t_*)^{\rho_{m+1}},\ldots,\xi_{n-1}=\lambda_{n-1}(t-t_*)^{\rho_{n-1}},\xi_n=(t-t_*)^{-1}\}$. The constants $\{\lambda_1,\ldots,\lambda_{n-1},t_*\}$ are arbitrary independent constants. Locally around t_*, the gradient of I can be written in terms of ξ_i:

$$\nabla I = J.\left(\frac{\partial I}{\partial \xi_1},\ldots,\frac{\partial I}{\partial \xi_n}\right) \tag{3.24}$$

with $J^{-1}=(\partial \xi_i/\partial x_j)$ is the Jacobian matrix. In terms of the variables ξ_i, the first integrals I, I' read

$$I=\sum_{i_j} K_{i_1,\ldots,i_n}\prod_{j=1}^{n} \xi_j^{i_j}, \tag{3.25}$$

$$I'=\sum_{i'_j} K'_{i'_1,\ldots,i'_n}\prod_{j=1}^{n} \xi_j^{i'_j}, \tag{3.26}$$

where the sum in the first (resp. second) integral is over all $\{i_j\}$ such that $i_1\rho_1+\cdots+i_{n-1}\rho_{n-1}+d\rho_n=0$ (resp. d') and the coefficients $K_{i_1,\ldots,i_n}\in\mathbb{C}$.

Using the relation (3.24) and the explicit form of ξ_i in terms of $(t-t_*)$, we obtain the gradients of I, I':

$$\nabla I = J.\sum_{i_j} L_{i_1,\ldots,i_n}(i_1(t-t_*)^{-\rho_1},\ldots,i_n(t-t_*)^{-\rho_n}), \tag{3.27}$$

$$\nabla I' = J.\sum_{i'_j} L'_{i'_1,\ldots,i'_n}(i'_1(t-t_*)^{-\rho_1},\ldots,i'_n(t-t_*)^{-\rho_n}). \tag{3.28}$$

Now, the relation $\alpha\nabla I+\alpha'\nabla I'=0$ implies $\alpha=\alpha'=0$. Therefore, written in terms of i_j, i'_j, there exists at least one pair of vector-integers $\{(i_1,\ldots,i_n),\ (i'_1,\ldots,i'_n)\}$ such that $\alpha(i_1,\ldots,i_n)+\alpha'(i'_1,\ldots,i'_n)=0\Rightarrow\alpha=\alpha'=0$. The proposition follows. □

As a corollary of our general result we obtain Yoshida's theorem:[32]

Corollary 3.6: If there exists at least one irrational or imaginary Kowalevskaya exponent, the system is not algebraically integrable.

Proof 3.6: From the previous proposition, we know that the existence of $(n-1)$ first integrals implies that there exists $(n-1)$ relations $N_i.\rho=d_i$, $i=1,\ldots,n-1$. Therefore, there exists a matrix $N\in \mathrm{GL}\ (n-1,\ \mathbb{Z})$, such that $N.\rho=d$ (ρ the vector of Kowalevskaya exponents, d the vector of degrees). We find $\rho=N^{-1}.d$ which implies $\rho\in\mathbb{Q}^{n-1}$. □

Yoshida's theorem gives necessary conditions for complete integrability. Conversely, we now find sufficient conditions for complete nonintegrability, that is, the nonexistence of at least one first integral:

Corollary 3.7: If all Kowalevskaya exponents are $\mathbb{Z}$-independent, then there is no rational first integral.

Proof 3.7: If $\rho_1,\ldots,\rho_n$ are $\mathbb{Z}$-independent, there is no relation $i_1\rho_1+\cdots+i_{n-1}\rho_{n-1}=d$ where $d\in\mathbb{Q}$. □

Corollary 3.8: If all Kowalevskaya exponents are $\mathbb{N}$-independent, then there is no polynomial first integral.

Proof 3.8: In the demonstration of the proposition we noticed that in the case of polynomial first integrals, the fundamental relation between the Kowalevskaya exponents and the degrees of first integrals reads $i_1\rho_1+\cdots+i_{n-1}\rho_{n-1}=d$, but with $i_j\in\mathbb{N}\forall j$. The result follows. □

These two last corollaries are equivalent to a recent result given in Ref. 20. Their results are obtained in a completely different setting and illustrated on many examples.

IV. ALGEBRAIC INTEGRABILITY FOR NONHOMOGENEOUS SYSTEMS

We found necessary conditions for a homogeneous system to be integrable. The conditions are simply given in terms of the Kowalevskaya exponents. More precisely, the maximum number of independent algebraic first integrals is given by the dimension of the vector space spanned by Kowalevskaya's exponents over the integers (the positive integers for polynomial first integrals). We can go one step further in our analysis. First, we focus on nonhomogeneous systems and we show that algebraic integrability brings only finite sheeting of the solution. That is, all solutions can be expanded in Puiseux series. In the first part, we show the absence of the logarithmic terms in the series expansions if the degrees of the first integrals are related to the Kowalevskaya exponents. This is reminiscent of the work of Ishii[37] for nth-order differential equations. Then, we show that the hypothesis on the Kowalevskaya exponents can be dropped and obtain the fundamental result that algebraic integrability always brings single-valuedness of the expansions.

A. Algebraic integrability and logarithmic branch points: Part I

We consider a system $S(f;x,t)$. We have seen in the previous sections that a necessary condition for algebraic integrability is that the set of all Kowalevskaya exponents $K_\alpha = \cup_{i=1}^n \{\rho_i\}$ is such that

$$K_\alpha \in \mathbb{Q}^n, \tag{4.1}$$

for all possible balances (α,p).

Therefore, we assume that S follows this assumption. Another assumption is required. We assume that the system S is completely integrable with $(n-1)$ polynomial first integrals. For each balance (α,p), the first integral I_i has a weighted degree d_i. Let the set of weighted degree $D_\alpha = \cup_{i=1}^n \{d_i\}$ be such that

$$D_\alpha = K_\alpha \tag{4.2}$$

for all balances (α,p).

That is, we assume that the degrees of the first integrals can be identified with the Kowalevskaya exponents. This assumption also implies that each balance (α,p) is a *principal balance*. That is, all the Kowalevskaya exponents but one are positive, and all solutions can be expanded in Puiseux series with exactly $(n-1)$ arbitrary constants. This assumption is fundamental for the rest of the analysis since it allows us to identify the arbitrary constants in the series expansion with the arbitrary constants of the first integrals:

Theorem 4.1: *Let the system $S:\dot{x} = f(x) \in QM(n)$ be algebraically integrable with $D_\alpha = K_\alpha \forall$ balances (α,p). Then, all solutions can be expanded in Puiseux series.*

In particular, if $K_\alpha \in \mathbb{Z}^n$, then the Painlevé test is satisfied in the variables $\{x,t\}$. The situation for which $K_\alpha \in \mathbb{Q}^n$ corresponds to the weak Painlevé case.[1] Let us already note that the converse statement is not true in general. They are, indeed, many systems with the Painlevé property which are not algebraically integrable (the Painlevé equations, for instance). It would be of much interest to find which extra conditions are required beside the Painlevé test for algebraic integrability to hold.

Proof 4.1: Consider the system $S(f(x);x,t)$ for given dominant balances (α,p) and assume first that matrix K is diagonalizable with a set of distinct eigenvalues $K_\alpha \in \mathbb{Q}^{\mathrm{n}}$. We have to show that the series expansions built on the dominant balance do not exhibit logarithmic branching. In other words, the compatibility conditions are satisfied for all Kowalevskaya's exponents. For all positive Kowalevskaya's exponents, we define $h_i = s\rho_i$ where $h_i \in \mathbb{N}$ and s is the smallest natural number for which such a relation holds for all i.

We want to show by recurrence that if the compatibility conditions are satisfied up to the Kowalevskaya exponent ρ', then they will be satisfied for the next Kowalevskaya exponent $\rho > \rho'$.

For the dominant balance (α,p), there exists a formal series expansion, solutions of $S(f(x);x,t)$:

$$x=(t-t_*)^g\sum_{i=1}^{\infty} a_i(t-t_*)^{i/s}, \tag{4.3}$$

where $a_i=a_i(\log(t-t_*))=\Sigma_j a_{ij}(\log(t-t_*))^j$ is a polynomial in $\log(t-t_*)$ of degree less than or equal to i.

If the compatibility conditions are satisfied up to $\rho'=h'/s$, then the coefficients a_i are independent of $\log(t-t_*)$:

$$a_i=a_{i0} \quad \forall i<h, \tag{4.4}$$

where $h=\rho s$. We have to prove that the existence of a first integral I of degree $d=\rho$ implies that a_h is also independent of $\log(t-t_*)$.

The most general form of a_h is $a_h=a_{h0}+a_{h1}\log(t-t_*)$. The recursion relation for the coefficient a_i $(i=1,\dots,\ h-1)$ reads

$$K.a_{i0}=\frac{i}{s}\,a_{i0}+P_{i0}(a_{10},\dots,a_{i-1,0}), \quad i=1,\dots,h-1, \tag{4.5}$$

where P_{i0} is polynomial in its arguments and can be obtained by the recursion relation for the coefficients a_{ij}.

Taking into account the possibility of a logarithmic contribution for a_h, the recursion relation gives rise to a linear system for a_{hj}, $j=0,\ 1$:

$$K.a_{h0}=\frac{h}{s}\,a_{h0}+P_h(a_1,\dots,a_{h-1})+a_{h1}, \tag{4.6}$$

$$K.a_{h1}=\frac{h}{s}\,a_{h1}. \tag{4.7}$$

The general solution of this system is

$$a_{h1}=\mu\beta_\rho, \tag{4.8}$$

$$a_{h0}=\lambda\beta_\rho+\delta_\rho, \tag{4.9}$$

where $\beta_\rho\in\mathbb{C}^n$ is the eigenvector of K of eigenvalue ρ, $\lambda\in\mathbb{C}$ is an arbitrary constant, and $\delta_\rho\in\mathbb{C}^n$ is a constant vector.

The constant μ is fixed by the compatibility condition:

$$\mu=\overline{\beta_\rho}.P_h, \tag{4.10}$$

where $\overline{\beta_\rho}$ is the eigenvector of K^T.

If the compatibility conditions at Kowalevskaya's exponent ρ are satisfied, then $\mu=0$.

Now, we consider the first integral $I(x)=C$, where C is an arbitrary constant and $I(x)$ is a function in many variables. This first integral is, by definition, constant along all solutions. Therefore, it is constant on the formal solution (4.3). We then expand the first integral in powers of $(t-t_*)$:

$$I\left[(t-t_*)^g\sum a_i(t-t_*)^{i/s}\right]=(t-t_*)^{-d}\sum b_i(t-t_*)^{i/s}, \tag{4.11}$$

where a_i and b_i are polynomial functions of $\log(t-t_*)$. Now, the integral is constant on any solution curve. Therefore, one has $b_i=0 \forall i \neq h$. To order $O((t-t_*)^0)$, the coefficient b_h can be estimated:

$$b_h=\nabla I(a_0).a_{h0}+\log(t-t_*)\nabla I(a_0).a_{h1}+O(\log(t-t_*)^2). \tag{4.12}$$

The general form of the coefficients a_{h0} and a_{h1} are known from (4.8). Therefore, a new arbitrary constant λ enters at order $O((t-t_*)^0)$. This constant has to match the arbitrary constant of the first integral:

$$\lambda \nabla I(a_0).\beta_\rho=C. \tag{4.13}$$

Now, consider the first logarithmic contribution to the integral $I=I(x)$, that is, the order $O(\log(t-t_*))$ of $I(x(t-t_*))$:

$$\mu \nabla I(a_0).\beta_\rho=0. \tag{4.14}$$

The constant C is arbitrary, therefore one has $\nabla I(a_0).\beta_\rho \neq 0$, and we conclude that $\mu=0$ and that the compatibility conditions for the Kowalevskaya exponent ρ is satisfied if the compatibility conditions up to Kowalevskaya's exponent ρ are satisfied. Iterating this process up to the last positive Kowalevskaya exponent, we see that there is no logarithmic contribution, that is, all solution can be expanded in Puiseux series. □

1. Problems and limitation

We have seen in this section that algebraic integrability is closely related to the absence of logarithmic points in the complex plane. This is a first step to a general relationship between integrability and singularity analysis. There is, however, a limitation in this method in the assumption on the degrees of first integrals, they should be related to the resonances of the system. If the degree d is not an eigenvalue of K, then from relation (4.13) with $C=0$, we have

$$\nabla I(a_0)=(0), \tag{4.15}$$

that is, the gradient of the first integral whose degree is not a Kowalevskaya exponent vanishes identically, and nothing can be said about the existence of logarithmic singularities in the complex plane.

This is the main difficulty of the method. For instance, suppose that there exists an irreducible first integral of degree $d>\rho_{\max}$, what can be said about the series expansions? Our proof relies heavily on the fact that $\nabla I(a_0) \neq (0)$, therefore in the case the gradient vanishes identically, no information can be found. Can we prove that such integrals do not exist, that is, if $K_\alpha \in \mathbb{Q}^n$, then one always has $D_\alpha=K_\alpha$?

We now overcome this difficulty by deriving necessary conditions for integrability of nonhomogeneous vector fields independently of the degree of the first integrals.

B. Algebraic integrability and logarithmic branch points: Part II

In the last section we proved that if the degrees of the first integrals are identical to the Kowalevskaya exponents, then algebraic integrability brings the single-valuedness of the solution. We now show that this result can be generalized in the sense that the assumption on the resonances can be dropped. Indeed, the key of the former proof is that the first logarithmic contribution to the first integral is given in terms of the gradient of the first integral around the particular solution $x=a_0 t^{-g}$. Therefore, the first integrals whose gradient vanishes on this particular solution cannot be used to prove single-valuedness.

We now extend the former result by considering variations around the general solution. We only retain from the previous assumptions that all balances are principal, that is, for each balance there exists $(n-1)$ positive Kowalevskaya exponents: Let $K_\alpha=\{\rho_1,..., \rho_{n-1}, -1\}$ be the set of Kowalevskaya's exponents for the balance (α,g). Then, for all balances (α,g),

$$\rho_i>0 \quad \forall \rho_i \in K_\alpha, \quad i=1,\ldots,n-1. \tag{4.16}$$

Theorem 4.2: *Assume that the system $S:\dot{x}=f(x)$ has only principal balances and is algebraic integrable with $(n-1)$ first integrals $I_1,\ldots,I_{n-1}$. Then, all solutions can be expanded in Puiseux series.*

Proof 4.2: Under the assumptions, there exists a formal expansion of the solution around a movable singularity t_* of the form:

$$x=\sum_{i=1}^{\infty} x_i Z^i, \tag{4.17}$$

$$Z=\log(t-t_*), \tag{4.18}$$

$$x_i=(t-t_*)^p \sum_{j=1}^{\infty} a_{ij}(t-t_*)^{j/s} \equiv (t-t_*)^p \Psi_i, \tag{4.19}$$

where $p\in\mathbb{Q}^n$ and $s\in\mathbb{N}$.

We consider a first integral I. It is constant on all solutions. Therefore, one has

$$I(x(t-t_*))=C=(t-t_*)^{-d}[I(\Psi_0)+Z\boldsymbol{\nabla} I(\Psi_0).\Psi_1+O(Z^2)]. \tag{4.20}$$

Since I is constant, we obtain to orders $O(Z^0)$, $O(Z^1)$:

$$(t-t_*)^{-d}I(\Psi_0)=C, \tag{4.21}$$

$$\boldsymbol{\nabla} I(\Psi_0).\Psi_1=0. \tag{4.22}$$

These relations hold for all first integrals I_i:

$$(t-t_*)^{-d_i}I_i(\Psi_0)=C_i, \quad i=1,\ldots,n-1, \tag{4.23}$$

$$\nabla_i(\Psi_0).\Psi_1=0, \quad i=1,\ldots,n-1. \tag{4.24}$$

We conclude from the first expression that the arbitrary constants $C_i=C_i(\lambda_1,\ldots,\lambda_{n-1})$ are polynomial in the arbitrary constants $(\lambda_1,..., \lambda_{n-1})$ appearing in the series Ψ_0. The second expression implies that Ψ_1 is proportional to Ψ_0:

$$\Psi_1=K(t)\Psi_0, \tag{4.25}$$

where $K(t)$ is analytic in t.

The series Ψ_1 can be locally expanded around t_*:

$$\Psi_1=\mu p(t-t_*)^q+O((t-t_*)^{q+1}), \tag{4.26}$$

where $q\in\mathbb{N}$ and $\mu\in\mathbb{C}$ is an arbitrary constant and p is the leading exponent of $x_0=c(t-t_*)^p(1+O((t-t_*)))$.

From the other point of view, we know from the local analysis [see Eq. (4.8)] that

$$\Psi_1=\mu\beta_\rho(t-t_*)^\rho+O((t-t_*)^{\rho+1}), \tag{4.27}$$

where ρ is the first Kowalevskaya exponent at which the compatibility conditions are not identically satisfied and μ is a constant fixed by the compatibility conditions; β_ρ is the eigenvector of K of eigenvalue ρ.

Therefore, we have that either $\mu=0$ or $p=\beta_\rho$, which is not possible since we know that $p.\beta_\rho=0$ for all positive Kowalevskaya exponents. We conclude that $\mu=0$ and $\Psi_1=0$.

Now, if $\Psi_1=0$, we obtain to order $O(Z^2)$

$$\nabla I(\Psi_0).\Psi_2=0, \tag{4.28}$$

and the same conclusions apply to Ψ_2.

Iterating this process, we conclude that $\Psi_i=0 \forall i>0$, that is, the solution can be expanded in Puiseux series. □

1. Example: The Lorenz system

As an example, we consider the famous Lorenz system[38] which has been thoroughly investigated as a dynamical system[39] and eventually became a paradigmatic system for integrability theories.[2,3,5,7,9,40,41] From the singularity analysis point of view, it was first studied by Segur[42] and more recently in Ref. 43. The system reads

$$\dot{x}=\sigma(y-x), \tag{4.29}$$

$$\dot{y}=\rho x-y-xz, \tag{4.30}$$

$$\dot{z}=xy-\beta z, \tag{4.31}$$

$$\tag{4.32}$$

where $x,y,z,\ \sigma,\ \beta,\ \rho\in\mathbb{R}$.

The Lorenz system has only principal balances with $c=(2i,-2i/\sigma,-2/\sigma)$ and $g=(1,1,2)$. The resonance set is $K_1=\{-1,2,4\}$.

There is one set of parameters values for which the system has two first integrals, namely $\{\beta,\sigma,\rho\}=\{1,\frac{1}{2},0\}$ with first integrals

$$I_1=(x^2-2\sigma z)e^t, \tag{4.33}$$

$$I_2=(y^2+z^2)e^{2t}, \tag{4.34}$$

and the solutions can be expressed in terms of Jacobi elliptic functions.

In addition, there exist two set of values for which the system satisfies the Painlevé test: $\{\beta,\sigma,\rho\}=\{1,2,\frac{1}{9}\}$ with one time-dependent integral

$$I_3=(x^2-2\sigma z)e^{2\sigma t}, \tag{4.35}$$

and $\{\beta,\sigma,\rho\}=\{0,\frac{1}{3},\rho\}$ with the time-dependent integral

$$I_4=(-\rho x^2+\tfrac{1}{3}y^2+\tfrac{2}{3}xy+x^2z-\tfrac{3}{4}x^4)e^{(4/3)t}. \tag{4.36}$$

For all other values of the parameters the system does not satisfy the Painlevé test and we conclude, using Theorem 3.2:

Proposition 4.3: If $\{\beta,\sigma,\rho\}\neq\{1,\frac{1}{2},0\}$, *the Lorenz system is not algebraically integrable (with two first integrals).*

2. A conjecture

We have seen that the existence of a complete set of first integrals and only principal balances constrains the system so that it must have the weak Painlevé property. Two questions are in order: Can we relax the conditions on the balances? Are these conditions also sufficient? In Ref. 25 it was suggested that it is indeed the case. Our conjecture reads then

If a system of ODEs with polynomial vector fields is algebraically integrable, then it enjoys the weak Painlevé property.

To date, among all known examples and to the best of my knowledge, there is no counterexample to this conjecture.

V. PARTIAL INTEGRABILITY

A. A natural arbitrary small parameter

We consider a nonhomogeneous system

$$S:\dot{x}=f(x), \quad x\in\mathbb{K}^n, \tag{5.1}$$

where f_i are rational functions of x over $\mathbb{K}$, a field of constants (typically $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$).

The problem is to find necessary conditions for the existence of first integrals. The theory we developed in the previous sections is only applicable for complete algebraic integrability. As a consequence, it cannot be directly applied to Hamiltonian systems, since for most of the Liouville integrable systems only half of the constants of motion are algebraic.

From the other point of view, the integrability conditions related to the Kowaleskaya exponents are based on similarity-invariant systems. However, most of the systems do not exhibit such a scaling property. Indeed, as soon as dissipation or damping is included under the form of linear terms, the system will lose the scale invariance. Nevertheless, similarity-invariant systems are the first-order systems in a perturbation expansion based on the scale invariance. With this idea in mind we can decompose the problem of finding necessary conditions for the existence of l first integrals $(l<n-1)$ into two parts.

The first part of the analysis consists of finding conditions for the existence of l first integrals of all weight-homogeneous parts of the vector field. According to the weight g, the vector field can be truncated so that

$$f(x)=f^{(0)}(x)+\cdots+f^{(m)}(x), \tag{5.2}$$

$$f^{(i)}(\alpha^{-g}x)=\alpha^{-g-1+i}f^{(i)}(x). \tag{5.3}$$

The leading weight-homogeneous system $\dot{x}=f^{(0)}(x)$ is scale invariant under the symmetry $(x\rightarrow\alpha^g x,\ t\rightarrow\alpha^{-1}t)$. According to this scaling symmetry, any first integral can be decomposed into a finite sum of weight-homogeneous components:

$$I(x,t)=I^{(0)}(x)+\epsilon I^{(1)}(x,t)+\cdots, \tag{5.4}$$

$$I^{(i)}(\alpha^g x)=\alpha^{d+i}I^{(i)}(x), \tag{5.5}$$

where $\epsilon=1/\alpha$.

The following lemma shows that a necessary condition for the existence of a first integral for a vector field is given by the existence of a first integral for its similarity invariant part.

Lemma 5.1: Let $I(x,t)$ be a first integral of $\dot{x}=f(x,t)$. Then, $I^{(0)}(x)$ is a first integral of $\dot{x}=f^{(0)}(x)$.

Proof 5.1: The condition for $I(x,t)$ to be a first integral reads

$$0=\nabla I.f+\partial_t I=\nabla I^{(0)}.f^{(0)}+\epsilon(\nabla I^{(0)}.f^{(1)}+\nabla I^{(1)}.f^{(0)}+\partial_t I^{(1)})+O(\epsilon^2). \tag{5.6}$$

This expansion is valid for arbitrary ϵ, indeed ϵ is an arbitrary parameter and can assume any value. Therefore, in the limit $\epsilon \to 0$, one finds $0 = \nabla I^{(0)} . f^{(0)}$. □

Let us stress again that ϵ is an arbitrary parameter that can assume any value and does not have to be small. Therefore, our method does not rely on the smallness of ϵ. If there is a small parameter in the system a perturbation expansion can be performed (see, for instance, Ref. 44).

The second part of the analysis is based on the existence of first integrals for the similarity invariant part. Assuming that such first integrals exist, we derive necessary conditions for the existence of first integrals for the nonhomogeneous contributions.
Our strategy is the following:

(1) Find necessary conditions for the existence of l $(0 < l < n)$ first integrals of the weight-homogeneous vector field $\dot{x} = f^{(0)}$.

(2) Build the l first integrals, $I_i^{(0)}$ $(i = 1, \ldots, l)$.

(3) Find necessary conditions for the existence of l' $(l' \leqslant l)$ first integrals for the nonhomogeneous systems.

(4) Build the l' first integrals F_i $(i = 1, \ldots, l')$. Each F_i is of the form

$$F = F^{(0)}(x) + \epsilon F^{(1)}(x,t) + \cdots , \tag{5.7}$$

$$F^{(0)} = P(I_1^{(0)}, \ldots, I_l^{(0)}), \tag{5.8}$$

where P is a rational, weight-homogeneous function.

The scaling symmetry is interesting in many respects. By contrast to the linear perturbation theory, starting from the lower linear terms of the systems and building higher-order corrections valid in the neighborhood of the fixed point, our nonlinear perturbation theory starts from the highest nonlinear terms and consider lower-orders perturbation. It allows us to build integrable corrections valid everywhere and for all values of the parameters.

B. Necessary conditions for partial integrability

Here we assume that the analysis has already been performed on the weight-homogeneous components of the vector fields and the explicit form of the first integrals are known. Therefore, we study the persistence of the first integrals, or homogeneous combinations of first integrals, when lower nonlinearities are added to the system.

We consider again the nonhomogeneous system

$$S : \dot{x} = f(x). \tag{5.9}$$

For this system, there may exist different truncations, that is, different weights g decomposing the vector field, according to the scaling $(x \to \alpha^g x,\ t \to \alpha^{-1} t)$, in dominant and nondominant parts. We consider all such vectors g and assume that each weight-homogeneous component admits the maximum number of possible first integrals. For a given vector g, we have a truncation of the vector field (5.2), so that $I_i = I_i(x)$ $(i = 1, \ldots, l)$ is a set of independent first integrals for the system $\dot{x} = f^{(0)}(x)$. Each first integral has a given weighted degree h_i w.r.t. the weight g:

$$I_i(\alpha^g x) = \alpha^{h_i} I_i(x). \tag{5.10}$$

We are interested in the existence of first integrals for the complete system. The most general form the first integrals can assume is given by

$$F(x,t) = F^{(0)}(x) + \epsilon F^{(1)}(x,t) + \epsilon^2 F^{(2)}(x,t) + \cdots , \tag{5.11}$$

where by construction $F^{(0)}$ is built on the first integrals I_i:

$$F^{(0)}=P(I_1,\dots,I_l), \tag{5.12}$$

where P is weight-homogeneous w.r.t. the scaling $(I_i\to\epsilon^{h_i}I_i)$.

Let us recall that two different first integrals F_1 and F_2 are independent iff $\nabla F_1^{(0)}$ and $\nabla F_2^{(0)}$ are linearly independent. Now, we derive necessary conditions for the existence of F.

First, we note that there is a series expansion involving l free constants (corresponding to the l free arbitrary constants) and we compute the following $\psi-\epsilon$ expansion:[44]

$$x=x_0+\epsilon x_1+\epsilon^2 x_2+\cdots, \tag{5.13}$$

$$x_i=\sum_{j=0}^{i-k+1} s_{ij}[\log(t-t_*)]^j, \tag{5.14}$$

where $s_{ij}\in\mathbb{C}((t-t_*))$ are convergent Laurent series with finite principal parts and k is the first order in ϵ where logarithmic corrections are required, so that x reads

$$x=s_{00}+\epsilon s_{10}+\epsilon^2 s_{20}+\cdots+\epsilon^k(s_{k0}+s_{k1}\log(t-t_*))+O(\epsilon^{k+1}). \tag{5.15}$$

Second, assuming there exists a first integral, we expand $F(x)$ around x_0:

$$\begin{aligned}F(x,t)=F(x_0)+\epsilon(\nabla F^{(0)}(x_0).x_1+F^{(1)}(x_0,t))+\epsilon^2(\nabla F^{(0)}(x_0).x_2\\ +\nabla F^{(1)}(x_0,t).x_1+\nabla^2F^{(0)}(x_0):x_1x_1+F^{(2)}(x_0,t))+O(\epsilon^3).\end{aligned} \tag{5.16}$$

Note that $F(x,t)$ is constant along all solution curves and the parameter ϵ is a scaling parameter which assume arbitrary values. As a consequence, each order in ϵ is constant in time. In particular, there is no logarithmic dependence and we obtain the following.

Lemma 5.2: Assume there exists a first integral $F(x,t)=F^{(0)}(x)+\epsilon F^{(1)}(x,t)+\cdots$ *for the system S. Then the following identity holds*:

$$\nabla F^{(0)}.s_{k1}=0. \tag{5.17}$$

Proof 5.2: Inserting the expansion of x in terms of s_{ij} in (5.16), we obtain

$$\begin{aligned}F(x,t)=F^{(0)}(s_{00})+\epsilon(\nabla F^{(0)}(s_{00}).s_{10}+F^{(1)}(s_{00},t))+\cdots\\ +\epsilon^k(\nabla F^{(0)}(s_{00}).(s_{k0}+\log(t-t_*)s_{k1}+\dots))+O(\epsilon^{k+1}).\end{aligned} \tag{5.18}$$

The coefficients of ϵ are constant at each order and the first logarithmic correction enters to order $O(e^k)$. Hence, we obtain

$$\nabla F^{(0)}.s_{k1}=0. \tag{5.19}$$

□

There is yet another instructive way to complete this result. To do so, we consider a system $S:\dot{x}=f(x)$ and assume it has a first integral $I(x,t)$. Let $\hat{x}=\hat{x}(t)$ be a solution of S and introduce the *variational equation*,

$$\dot{v}=Df(\hat{x}).v, \tag{5.20}$$

where $Df(\hat{x})$ is the Jacobian matrix estimated on the solution $\hat{x}$.

Lemma 5.3 (Poincaré): Let $I=I(x)$ *be a first integral of S and let v be any solution of the variational equation. Then,*

$$\nabla I(\hat{x}).v=C, \tag{5.21}$$

where C is a constant.

Now, consider $F^{(0)}$, a first integral of $\dot{x}=f^{(0)}(x)$. The solution s_{00} is a local convergent Laurent series around the singularity t_*, and s_{k1} is, by construction, a solution of

$$\dot{v}=Df(s_{00}).v. \tag{5.22}$$

Therefore, using Poincaré's lemma, we obtain $\nabla F^{(0)}.s_{k1}=C$. Inserting this result in (5.18) and asking the log contribution to vanish identically at each order in ϵ, we obtain Lemma 5.2.

The point of this lemma is to provide a necessary condition for $F^{(0)}$ to be the homogeneous part of the nonhomogeneous first integral $F(x,t)$. However, the functional form of $F^{(0)}$ is not known except for the fact that it is built out of the first integrals I_i $(i=1,\ldots,l)$.

Now, consider the l first integrals $I_i=I_i(x)$ $(i=1,\ldots,l)$. For each first integral, we define the conditions

$$c_i=\nabla I_i^{(0)}(s_{00}).s_{k1}. \tag{5.23}$$

If there exist l first integrals for the system S, they are built on the first integrals I_i, and the existence condition for these first integrals depends on the conditions c_i:

Proposition 5.4: Assume there exist l independent analytic first integrals $I_i=I_i(x)$ for the similarity invariant system $S_0:\dot{x}=f^{(0)}(x)$. Then, a necessary condition for the existence of l independent analytic first integrals for $\dot{x}=f(x)$ is $c_1=c_2=\cdots=c_l=0$.

Proof 5.4: Suppose that there exist l independent first integrals F_i for S. Let F be any of such integral. According to Lemma 5.3, for each integral we have

$$\nabla F^{(0)}(s_{00}).s_{k1}=0. \tag{5.24}$$

As already explained, $F^{(0)}$ is weight-homogeneous in I_j so that

$$\nabla F^{(0)}(s_{00})=\sum_{j=1}^{l} A_{ij}\nabla I_j^{(0)}(s_{00}) \tag{5.25}$$

with $A\in \mathrm{GL}(l,\mathbb{C})$, owing to the independence of the first integrals and the fact that the series s_{00} depends on l free parameters.

Therefore the integrability condition reads

$$\sum_{j=1}^{l} A_{ij}\nabla I_j^{(0)}(s_{00}).s_{k1}=\sum_{j=1}^{l} A_{ij}c_j=0. \tag{5.26}$$

Since $A\in \mathrm{GL}(l,\mathbb{C})$, we have $c_i=0\,\forall i=1,\ldots,l$. □

1. A first example

As a first example, we consider the following three degrees of freedom Hamiltonian:[45]

$$H=\tfrac{1}{2}(p_1^2+p_2^2+p_3^2)+\tfrac{1}{4}(x_1^4+x_2^4+x_3^4)+\epsilon(\mu_1x_2x_3+\mu_2x_3x_1+\mu_3x_1x_2), \tag{5.27}$$

where $\mu_i\neq 0$, $i=1,2,3$

For $\epsilon=0$, the system is integrable with three obvious constants of motion, the Hamiltonians of the decoupled systems

$$I_i^{(0)}=2p_i^2+x_i^4, \quad i=1,2,3, \tag{5.28}$$

and the solutions can be expanded in Laurent series $s=(x_1,x_2,x_3,p_1,p_2,p_3)$:

$$s_{00}=(t-t_*)^p\sum_{i=0}^{\infty} a_i(t-t_*)^i, \tag{5.29}$$

where $p=(-1,-1,-1,-2,-2,-2)$, $a_i\in\mathbb{C}^6$, and a_4 is arbitrary (resonance $r\in K_1=\{-1,-1,-1,4,4,4\}$).

The series expansion for the perturbed problem can be readily computed:

$$x=s_{00}+\epsilon s_{10}+\epsilon^2(s_{20}+s_{21}\log(t-t_*))+O(e^3), \tag{5.30}$$

where $s_{ij}\in\mathbb{C}^6((t-t_*))$.

The first logarithmic contribution enters to order $O(\epsilon^2)$. The integrability conditions read

$$c_i=\nabla I_i^{(0)}.s_{21}, \quad i=1,2,3. \tag{5.31}$$

The conditions $c_1=c_2=c_3=0$ gives then

$$2\mu_2\mu_3-\mu_1\mu_3-\mu_1\mu_2=0, \tag{5.32}$$

$$-\mu_2\mu_3+2\mu_1\mu_3-\mu_1\mu_2=0, \tag{5.33}$$

$$-\mu_2\mu_3-\mu_1\mu_3+2\mu_1\mu_2=0. \tag{5.34}$$

As a consequence, if $\mu_i\neq\mu_j$, $i\neq j$, the Hamiltonian is not Liouville integrable, that is there exists at most another constant of motion beside the Hamiltonian. In Ref. 45, it was shown that the Hamiltonian (5.27) for $\mu_1=0$, $\mu_2=\mu_3=1$, does not possess a second constant of motion.

2. Another example

The second example is also a three degrees of freedom Hamiltonian:[34]

$$H=\tfrac{1}{2}(p_1^2+p_2^2+p_3^2)+(x_1x_3^2+x_2x_3^2)+\epsilon(\mu_1x_1^2+\mu_2x_2^2+\mu_3x_3^2), \tag{5.35}$$

where $\mu_i\neq 0$, $i=1,2,3$.

We first consider the unperturbed system ($\epsilon=0$). Is the system Liouville integrable? By inspection or using direct methods, a second constant of motion is easily found:

$$I=p_1-p_2. \tag{5.36}$$

A third constant of motion is lacking to complete the integration:

Lemma 5.5: The Hamiltonian system $H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+(x_1x_3^2+x_2x_3^2)$ *is not Liouville integrable.*

Proof 5.5: We compute the Kowalevskaya exponents for the scale-invariant solution $(x_1,x_2,x_3,p_1,p_2,p_3)=-3t^{-3}(t/2,t/2,-t,-1,-1,-2)$: $\rho\in\{-1,2,3,6,(5\pm i\sqrt{23})/2\}$. The occurrence of irrational Kowalevskaya exponents are incompatible with Liouville integrability for potential with diagonal kinetic part (see Theorem 3.4). □

Now, we consider the full system (5.35), and build the ψ–ϵ expansion up to order $O(\epsilon)$:

$$x=s_{00}+\epsilon(s_{10}+s_{11}\log(t-t_*))+O(\epsilon^2), \tag{5.37}$$

where $x=(x_1,x_2,x_3,p_1,p_2,p_3)$ and s_{00},s_{10} are Laurent series. The series s_{11} reads

$$s_{11}=-\tfrac{3}{4}(\mu_1-\mu_2)(1,-1,0,0,0,0)+O((t-t_*)^2). \tag{5.38}$$

The conditions $0=\nabla H.s_{11}$ and $0=\nabla I.s_{11}$ lead to $\mu_1=\mu_2$. In this case the second constant of motion is found to be

$$I=(p_1^2+\mu_1 x_1^2)+(p_2^2+\mu_1 x_2^2)-2(p_1 p_2+\mu_1 x_1 x_2). \tag{5.39}$$

We have proved the following.

Proposition 5.6: The Hamiltonian system (5.35) does not have a second constant of motion unless $\mu_1=\mu_2$.

VI. CONCLUSIONS

This paper studies the relationship between the singularity analysis and the algebraic integrability. We first considered weight-homogeneous systems and showed that there exists a fundamental relationship between the Kowalevskaya exponents and the degrees of the first integrals. We concluded from this relation that the number of first integrals is equal or less than the dimension of the vector space spanned by the Kowalevskaya exponents over the integers. I believe that this relationship, although restricted, is the most complete information one can hope to obtain by using only the Kowalevskaya exponents. In order to obtain more information on the integrability, or lack thereof, of a system, one has to take into account the specific structure of logarithmic or algebraic branch points. As a corollary, we proved the well-known Yoshida's statement: a necessary condition for complete algebraic integrability is that all Kowalevskaya exponents be rational. A second important result concerns completely algebraically integrable system. We proved that if all balances are principal [i.e., $(n-1)$ positive resonances], then algebraic integrability implies that all solutions can be expanded in Puiseux series. Moreover, I believe that the assumption that all balances are principal can actually be removed. The third main result of this paper concerns partial integrability. Partial integrability seems the most common feature of dynamical systems emerging from physical model; indeed often by symmetry or conservation laws, a few constants of the motion exist. Assuming that the weight-homogeneous part of the vector field is algebraically integrable, we derived necessary conditions for the existence of first integrals based on the non-existence of logarithmic branch points. This result can be used, for instance, to show that first integrals disappear in the presence of linear dissipative terms.

I hope that these three aspects put together will give a general picture of the type of information concerning integrability one can hope to obtain using singularity analysis. It also provides direct algorithmic methods to answer such questions.

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[1] A. Ramani, B. Grammaticos, and T. Bountis, ''The Painlevé property and singularity analysis of integrable and non-integrable systems,'' Phys. Rep. **180**, 159–245 (1989).

[2] F. Schwarz, ''An algorithm for determining polynomial first integrals of autonomous systems of ordinary differential equations,'' J. Symbol. Comput. **1**, 229–233 (1985).

[3] F. Schwarz, ''Existence theorems for polynomial first integrals,'' in *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, Bonn, Germany, edited by S. M. Watt (ACM, New York, 1991).

[4] F. Schwarz and W. H. Steeb, ''Symmetries and first integrals for dissipative systems,'' J. Phys. A **17**, L819–L823 (1984).

[5] H. J. Giacomini, C. E. Repetto, and O. P. Zandron, ''Integrals of motion for three-dimensional non-Hamiltonian dynamical systems,'' J. Phys. A **24**, 4567–4574 (1991).

[6] T. Carleman, ''Applications de la théorie des équations intégrales linéaires aux systèmes d'équations différentielles nonlinéaires,'' Arkiv. Math. **228**, 63–87 (1931).

[7] M. Kús, ''Integrals of motion for the Lorenz system,'' J. Phys. A **16**, L689–L691 (1983).

[8] W. H. Steeb, ''A note on Carleman linearization,'' Phys. Lett. A **140**, 336–338 (1989).

[9] A. Goriely, *Transformations Quasi-Monomiales et Intégrabilité* (Mémoire de licence, Bruxelles, 1989).

[10] M. Codutti, *Nodes: Nonlinear Ordinary Differential Equations Solver* (Mémoire de Licence, Bruxelles, 1990).

[11] S. L. Ziglin, ''Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics,'' Funct. Anal. Appl. **16**, 181–189 (1983).
[12] S. L. Ziglin, ''Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics II,'' Funct. Anal. Appl. **17**, 6–17 (1983).
[13] H. Yoshida, ''A criterion for the non-existence of an additional integral in Hamiltonian systems with homogeneous potential,'' Phys. D **29**, 128–142 (1987).
[14] H. Yoshida, Phys. Lett. A **141**, 108–112 (1989).
[15] H. Yoshida, B. Grammaticos, and A. Ramani, ''Painlevé resonances versus Kowalevski exponents: Some exact results on singularity structure and integrability of dynamical systems,'' Acta Apl. Math. **8**, 75–103 (1987).
[16] H. Yoshida, ''Non-integrability of the truncated Toda lattice Hamiltonian at any order,'' Commun. Math. Phys. **116**, 529–538 (1988).
[17] H. Yoshida, ''Non-integrability of a class of perturbed Kepler problems in two dimensions,'' Phys. Lett. A **120**, 388–390 (1987).
[18] Y. Itoh, ''Integrals of a Lotka-Volterra system of odd number of variables,'' Prog. Theor. Phys. **78**, 507–510 (1987).
[19] S. Itchiaroglou, ''On the nonintegrability of Yang-Mills potentials,'' J. Phys. A. **22**, 3461–3465 (1989).
[20] J. Moulin-Ollagnier, A. Nowicki, and J. M Strelcyn, ''On the nonexistence of constants of derivations: the proof of a theorem of Jouanolou and its development,'' Preprint, 1994.
[21] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw–Hill, New York, 1955).
[22] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J. M Strelcyn, and S. Wojciechowski, ''Integrals of quadratic ordinary differential equations in $\mathbb{R}^3$: The Lotka-Volterra system,'' Phys. A **163**, 683 (1989).
[23] M. Kruskal and P. A. Clarkson, ''The Painlevé-Kowalesvski and poly-Painlevé test for integrability,'' Stud. Appl. Math. **86**, 87–165 (1992).
[24] M. Adler and P. van Moerbeke, ''Algebraic completely integrable systems: a systematic approach,'' Séminaire de Mathématique, Institut de Mathématique pure et appliquée, Université Catholique de Louvain **1**, 165–309 (1987).
[25] N. Ercolani and E. Siggia, ''Painlevé property and geometry,'' Phys. D **34**, 3303–3346 (1989).
[26] A. R. Forsyth, *Theory of Differential Equations* (Cambridge U.P., Cambridge, 1900).
[27] M. J. Prelle and M. F. Singer, ''Elementary first integrals of differential equations,'' Trans. Am. Math. Soc. **279**, 215–229 (1983).
[28] R. Shtokhamer, ''Solving first order differential equations using the Prelle-Singer algorithm,'' Technical Report 88-09, Center For Mathematical Computation, University of Delaware, Vol. 9, pp. 1–18, 1988.
[29] R. Shtokhamer, N. Glinos, and B. F. Caviness, ''Computing elementary first integrals of differential equations,'' Comput. Math. Conf. (Stanford) **1**, 1–9 (1986).
[30] M. Ishii, ''Many-valuedness of first integrals, their domains and many valued solutions,'' Progr. Theor. Phys. **87**, 93–101 (1992).
[31] H. Yoshida, ''Necessary conditions for the existence of algebraic first integrals I,'' Celest. Mech. **31**, 363–379 (1983).
[32] H. Yoshida, ''Necessary conditions for the existence of algebraic first integrals II,'' Celest. Mech. **31**, 381–399 (1983).
[33] S. Kowalevski, ''Sur le problème de la rotation d'un corps solide autour d'un point fixe,'' Acta Math. **12**, 177–232 (1889).
[34] J. Hietarinta, ''Direct methods for the search of the second invariant,'' Phys. Rep. **147**, 87–154 (1987).
[35] P. Lochak, ''Pairing of the Kowaleska exponents in Hamiltonian systems,'' Phys. Lett. A **108**, 188–190 (1985).
[36] M. Kummer, R. C. Churchill, and D. L. Rod, ''On Kowaleskaya exponents,'' in *Essays on Classical and Quantum Mechanics*, edited by J. Ellison and H. Uberall (Gordon and Breach, New York, 1991).
[37] M. Ishii, ''Painlevé property and algebraic integrability of single variable ordinary differential equations with dominants,'' Prog. Theor. Phys. **84**, 386–391 (1990).
[38] E. N. Lorenz, ''Deterministic nonperiodic flow,'' J. Atmospher. Sci. **20**, 130–141 (1963).
[39] C. Sparrow, *The Lorenz Equations* (Springer-Verlag, New York, 1982).
[40] W. H. Steeb, ''Continuous symmetries of the Lorenz model and the Rikitake two-disc dynamo system,'' J. Phys. A **15**, L389–L390 (1982).
[41] J. M. Strelcyn and S. Wojciechowski, ''A method of finding integrals for three-dimensional dynamical systems,'' Phys. Lett. A **133**, 207–212 (1988).
[42] H. Segur, ''Soliton and the inverse scattering transform,'' in *Topics in Ocean Physics*, edited by A. R. Osborne and P. Malanotte Rizzoli (North–Holland, Amsterdam, 1982), pp. 235–277.
[43] G. Levine and M. Tabor, ''Integrating the nonintegrable: Analytic structure of the Lorenz system revisited,'' Phys. D **33**, 189–210 (1988).
[44] A. Goriely and M. Tabor, ''The singularity analysis for nearly integrable systems: homoclinic intersections and local multivaluedness,'' Phys. D **85**, 93–125 (1995).
[45] E. Meletlidou and S. Ichtiaroglou, ''On the number of isolating integrals in perturbed Hamiltonian systems with $n \geqslant 3$ degrees of freedom,'' Preprint, 1993.