ELSEVIER

Physica D 105 (1997) 45–61

PHYSICA D

Nonlinear dynamics of filaments II. Nonlinear analysis

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Received 12 April 1996; accepted 4 December 1996
Communicated by C.K.R.T. Jones

Abstract

The linear stability analysis of the full time-dependent Kirchhoff equations for elastic filaments gives precise information about possible dynamical instabilities. The associated dispersion relations derived in the preceding paper provides the selection mechanism for the shapes selected by highly unstable filaments. Here we perform a nonlinear analysis and derive new amplitude equations which describe the dynamics above the instability threshold. The straight filament is studied in detail and the motion is shown to be described by a pair of nonlinear Klein–Gordon equations which couple the local deformation amplitude to the twist density. Of particular interest is the effect of boundary conditions on the instability threshold. It is shown that with suitable choice of boundary conditions the threshold of instability is delayed. We also show the existence of pulse-like and front-like traveling wave solutions.

Keywords: Elasticity; Kirchhoff equations; Amplitude equations

1. Introduction

The study of thin rods has a long tradition in mechanics and engineering dating back to Euler and Lagrange. In recent years the study of filamentary structures has played an increasingly important role for the description of many phenomena appearing in physical, biological and chemical systems [1–14].

The Kirchhoff equations provide a well-established model to describe the dynamics of thin filaments within the approximations of linear elasticity theory. In a companion paper [15], we studied the stability of stationary solutions which describe finite (or infinite) rods in an equilibrium state. The parameters controlling the stability of these solutions are often taken to be the twist and/or the tension at the ends. In the preceding paper, we developed a perturbation scheme at the level of the local orthonormal basis (the director basis) attached to the rod axis. We derived the general linear equations controlling the stability of the given stationary solutions. In particular we fully described the instability of the twisted planar ring and the straight rod. The dispersion relations relating spatial structure to growth rates in time were derived in these cases and different situations were considered.

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PII S0167-2789(96)00003-1

Although the linear analysis provides a valuable picture of the different unstable solutions and the bifurcation points in parameter space, it does not give the behavior of the system beyond bifurcation. The usual approach to studying this stage of the motion is to derive nonlinear amplitude equations. This *weakly nonlinear analysis* is performed by introducing new, longer, scales for the independent variables and allowing the amplitudes of the linear modes to vary on these scales. These methods are well-known in different physical settings such as the modeling of convection phenomena in fluid mechanics for which they were first derived [16]. In these cases the dissipative character of the system plays an important role in singling out the excitable modes in the system. Many other physical systems have been studied along the same lines (for an exhaustive review of this approach see [17,18]). In the domain of elasticity, there have been several attempts to apply these methods. However most, if not all, of this work has been limited to the beam equation [19–22] which is a single fourth-order differential equation giving a phenomenological description of the deformation of a beam under tension and compression. However, it lacks the crucial physical characteristics of twist and three-dimensionality. To the best of our knowledge a nonlinear analysis of the full time-dependent Kirchhoff equations has never been performed. We show here that the problem of localization can only be fully understood in this framework.

Rather than considering the general problem, we illustrate our approach with one of the simplest physical system: the twisted straight rod under tension. The control parameters, twist and tension, are chosen in such a way that the system is very close to, but beyond, the bifurcation point where the straight solutions become unstable. The linear analysis of this problem has a long tradition dating back to Euler and is one of the classical examples of bifurcation in elastic media (see for instance [23–25]). However, a complete picture of the dynamical instabilities and the post-buckling behavior of this system is still lacking. Recent experiments show the persistence of some helical solutions after bifurcation followed by a dynamical jump to localized buckled solutions [26]. Close to the bifurcation point, a system of two coupled nonlinear Klein–Gordon equations for the amplitudes of the linear modes can be derived. These equations describe the dynamical behavior of the rod after bifurcation.

The paper is organized as follows: In Section 2, we recall the basic elements of curve dynamics and the Kirchhoff model in the director basis. In Section 3, we describe the perturbation expansion in this basis which was developed in the previous paper [15]. In Section 4, we perform the nonlinear analysis for the straight rod, derive the amplitude equations and study different particular solutions and different types of boundary conditions. In Section 5, we compare our results with the results obtained for the beam equation.

2. The Kirchhoff model

The Kirchhoff model describes the space and time evolution of thin filaments. In the thin filament approximation all physically relevant quantities characterizing the three-dimensional elastic body are attached to the central axis of the filament. We first review the relevant kinematics of space curves in three dimensions before proceeding with the Kirchhoff model.

Let $x = x(s,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ be a space curve, parametrized by the arc length s, whose position also depends on the time t. We assume that the map x is of at least class C^3. For each value of s and t, a director basis $\{d_1, d_2, d_3\}$ can be attached to the curve as follows: The vector $d_3(s,t) = x'(s,t)$ is the tangent vector of x at s (the prime denotes the s-derivative). The vectors $\{d_1(s,t), d_2(s,t)\}$ are two differentiable functions orthonormal to d_3 defined in such a way that $\{d_1, d_2, d_3\}$ forms, a right-handed orthonormal triad ($d_1 \times d_2 = d_3$, $d_2 \times d_3 = d_1$). In the case where d_1 is along d_3', the director basis specializes to the well-known Frenet triad where d_1 is the normal vector and d_2 the bi-normal vector. Given the director basis $\{d_1, d_2, d_3\}$, the curve can be reconstructed for all time and space by integrating the tangent vector, i.e. $x(s,t) = \int^s d_3(s,t)\,ds$.

The equations of motion describing the evolution of thin filaments can be easily written in terms of the director basis. Taking into account the orthonormality of the basis, one readily obtains the basis evolution with respect to arc length and time; namely

$$d_i' = \sum_{j=1}^{3} K_{ij} d_j, \quad i = 1, 2, 3, \tag{1a}$$

$$\dot{d}_i = \sum_{j=1}^{3} W_{ij} d_j, \quad i = 1, 2, 3, \tag{1b}$$

where $(\dot{\ })$ stands for the time derivative. W and K are the antisymmetric 3×3 matrices:

$$K = \begin{pmatrix} 0 & \kappa_3 & -\kappa_2 \\ -\kappa_3 & 0 & \kappa_1 \\ \kappa_2 & -\kappa_1 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{2}$$

The elements of K and W make up the components of the *twist and spin vectors*, namely $\kappa = \sum_{i=1}^{3} \kappa_i d_i$ and $\omega = \sum_{i=1}^{3} \omega_i d_i$. The compatibility condition between the two linear systems (1a) and (1b) is

$$W' - \dot{K} = [W, K], \tag{3}$$

where $[\cdot, \cdot]$ is the matrix commutator: $[W, K] = W \cdot K - K \cdot W$.

We now consider a naturally straight elastic rod with circular cross-section. The central axis is a space curve $x = x(s, t)$. The theory of thin rods gives the dynamics of rods whose length is much greater than the radius and whose curvature is sufficiently large relative to the small length scales in the problem. Within these approximations, a one-dimensional theory can be derived in which all the relevant physical quantities are averaged over the cross-sections and attached to the central axis. It follows that the total force $F = F(s, t)$ and the total moment $M = M(s, t)$ can be expressed locally in terms of the basis, i.e. $F = \sum_{i=1}^{3} f_i d_i$, $M = \sum_{i=1}^{3} M_i d_i$. The conservation of linear and angular momentum then leads to equations for the force and the moment [27]:

$$F'' = \rho A \ddot{d}_3, \tag{4a}$$

$$M' + d_3 \times F = \rho I (d_1 \times \ddot{d}_1 + d_2 \times \ddot{d}_2), \tag{4b}$$

where I is the moment of inertia (about a radial cross-section), ρ the density and A is the area of a (circular) cross-section.

These equations are closed by the constitutive relationships of the *linear* theory of elasticity:

$$M = EI[(\kappa_1 - \kappa_1^{(u)}) d_1 + (\kappa_2 - \kappa_2^{(u)}) d_2] + 2\mu I (\kappa_3 - \kappa_3^{(u)}) d_3, \tag{5}$$

where E is the Young modulus and μ the shear modulus and the $\kappa_i^{(u)}$ are the "intrinsic curvatures" which for the rest of the paper will be set to zero.

After the rescalings

$$\begin{aligned} &t \to t\sqrt{I\rho/AE}, && s \to s\sqrt{I/A}, \\ &F \to AEF, && M \to ME\sqrt{AI}, \\ &\kappa \to \kappa\sqrt{A/I}, && \omega \to \omega\sqrt{AE/I\rho}. \end{aligned} \tag{6}$$

Eqs. (4) and (5) take the form:

$$F'' = \ddot{d}_3, \tag{7a}$$
$$M' + d_3 \times F = d_1 \times \ddot{d}_1 + d_2 \times \ddot{d}_2, \tag{7b}$$
$$M = \kappa_1 d_1 + \kappa_2 d_2 + \Gamma \kappa_3 d_3, \tag{7c}$$

where $\Gamma = 2\mu/E = 1/(1+\sigma)$. The parameter Γ characterizes fully the elastic property of the filament, Γ varies between $\frac{2}{3}$ (incompressible case) and 1 (hyper-elastic case).

The relationship (7c) can be used to replace M in (7b). Together with the twist and spin equations (1a) and (1b), one obtains a system of nine equations (the Kirchhoff equations) for nine unknowns (f, κ, ω) which we write in the shorthand form

$$E(f, \kappa, \omega; s, t) = 0. \tag{8}$$

If one fixes a direction in space, the local vectors κ, ω can be expressed in terms of the Euler angles and the system (8) can then be written as a set of six equations for six unknowns. However, it was shown in [15] that the Euler angles are not the best choice of variables for developing a perturbation scheme to study the stability of stationary solutions and far more progress can be made by expanding all variables in terms of the director basis. The Kirchhoff equations (7) describe the time evolution of a rod with circular cross-section in the limit of small deformations where tractions on the external surfaces are assumed to vanish. For the rest of this paper a *Kirchhoff rod* or simply a *rod* is defined as a solution of this set of equations, (8), with proper boundary conditions and initial data.

3. Perturbation scheme

In order to study the stability of stationary solutions we develop a perturbation scheme at the level of the director basis. The idea is to expand the basis around the unperturbed stationary basis and require that it remains orthonormal at each order in the perturbation parameter, namely

$$d_i = d_i^{(0)} + \epsilon d_i^{(1)} + \epsilon^2 d_i^{(2)} + \cdots, \qquad i = 1, 2, 3. \tag{9}$$

The orthonormality condition $d_i \cdot d_j = \delta_{ij}$ leads to an expression for the perturbed basis in terms of the unperturbed basis:

$$d_i^{(1)} = \sum_{j=1}^{3} A_{ij}^{(1)} d_j^{(0)}, \tag{10a}$$
$$d_i^{(2)} = \sum_{j=1}^{3} (A_{ij}^{(2)} + S_{ij}^{(2)}) d_j^{(0)}, \tag{10b}$$
$$\vdots$$
$$d_i^{(n)} = \sum_{j=1}^{3} (A_{ij}^{(n)} + S_{ij}^{(n)}) d_j^{(0)}, \tag{10c}$$

where $A^{(k)}$ is the *antisymmetric* matrix:

$$A^{(k)} = \begin{pmatrix} 0 & \alpha_3^{(k)} & -\alpha_2^{(k)} \\ -\alpha_3^{(k)} & 0 & \alpha_1^{(k)} \\ \alpha_2^{(k)} & -\alpha_1^{(k)} & 0 \end{pmatrix} \tag{11}$$

and $S^{(k)}$ is a *symmetric* matrix whose entries depend only on $\alpha_i^{(j)}$ with $j < k$. For example,

$$S^{(2)} = \begin{pmatrix} -\frac{1}{2}(\alpha_2^{(1)})^2 - \frac{1}{2}(\alpha_3^{(1)})^2 & \frac{1}{2}\alpha_1^{(1)}\alpha_2^{(1)} & \frac{1}{2}\alpha_1^{(1)}\alpha_3^{(1)} \\ \frac{1}{2}\alpha_1^{(1)}\alpha_2^{(1)} & -\frac{1}{2}(\alpha_3^{(1)})^2 - \frac{1}{2}(\alpha_1^{(1)})^2 & \frac{1}{2}\alpha_2^{(1)}\alpha_3^{(1)} \\ \frac{1}{2}\alpha_1^{(1)}\alpha_3^{(1)} & \frac{1}{2}\alpha_2^{(1)}\alpha_3^{(1)} & -\frac{1}{2}(\alpha_1^{(1)})^2 - \frac{1}{2}(\alpha_2^{(1)})^2 \end{pmatrix}. \tag{12}$$

Once the vector $\alpha^{(1)}$ is known, it is an easy matter to reconstruct the perturbed rod by integrating the tangent vector

$$x(s,t) = \int^{s} \mathrm{d}s(d_3^{(0)} + \epsilon(\alpha_2^{(1)}d_1^{(0)} - \alpha_1^{(1)}d_2^{(0)})) + \mathrm{O}(\epsilon^2). \tag{13}$$

Any local vector $V = \sum_{i=1}^{3} v_i d_i$ can be expanded in terms of the perturbed basis, namely $V = V^{(0)} + \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \cdots$, where

$$V^{(1)} = \sum_i (v_i^{(1)} + (A^{(1)} \cdot v^{(0)})_i) d_i^{(0)}. \tag{14}$$

We can now express the first-order perturbation of the twist and spin matrix, i.e. $K = K^{(0)} + \epsilon K^{(1)} + \cdots$, $W = W^{(0)} + \epsilon W^{(1)} + \cdots$, where

$$K^{(1)} = \frac{\partial}{\partial s}A^{(1)} + [A^{(1)}, K^{(0)}], \tag{15a}$$

$$W^{(1)} = \frac{\partial}{\partial t}A^{(1)} + [A^{(1)}, W^{(0)}]. \tag{15b}$$

In this way, the higher-order perturbations can easily be obtained in terms of the lower-order terms.

Using these equations, one can write the first-order perturbation of Newton's equation (7a) and moment equation (7b) in terms of $(\alpha^{(1)}, f^{(1)})$. This system will be referred to as the *dynamical variational equations*. Its solutions control the stability, or lack thereof, of the stationary solutions with respect to linear time-dependent modes. To emphasize the linear character of these equations we rewrite them as a linear system of six equations for the six-dimensional vector $\mu^{(1)} = \{\alpha^{(1)}, f^{(1)}\}$:

$$L_E(\kappa^{(0)}, f^{(0)}) \cdot \mu^{(1)} = 0, \tag{16}$$

where L_E is a second-order differential operator in s and t whose coefficients depend on s through the unperturbed solution $\kappa^{(0)}, f^{(0)}$.

4. Nonlinear analysis

The procedure involved in developing an amplitude expansion is rather general. However some features of the analysis rely heavily on the specific form of the dispersion relation and the stationary solution. Therefore, for the sake of clarity, we perform the nonlinear analysis of the Kirchhoff model on the simplest stationary solutions: the twisted straight rod under tension. The linear analysis of the straight rod has been performed in [15]. We recall here the main ingredients before proceeding with the nonlinear analysis.

4.1. The linear analysis for the straight rod

We consider a straight twisted rod under a tension P^2. The total twist in the rod is γ. The stationary solutions are

$$\kappa^{(0)} = (0, 0, \gamma), \tag{17a}$$
$$f^{(0)} = (0, 0, P^2). \tag{17b}$$

The linearized system (16) is given by:

$$-\ddot{\mu}_2 + 2\gamma\mu_5' + \mu_4'' + P^2\mu_2'' - 2P^2\gamma\mu_1' - P^2\gamma^2\mu_2 - \mu_4\gamma^2 = 0, \tag{18a}$$
$$-2P^2\gamma\mu_2' - P^2\mu_1'' - \mu_5\gamma^2 + \ddot{\mu}_1 + P^2\gamma^2\mu_1 - 2\gamma\mu_4' + \mu_5'' = 0, \tag{18b}$$
$$\mu_6'' = 0, \tag{18c}$$
$$2\gamma\mu_2' + \mu_1'' - \gamma^2\mu_1 - \ddot{\mu}_1 - \Gamma\gamma\mu_2' + \Gamma\gamma^2\mu_1 - \mu_5 = 0, \tag{18d}$$
$$\mu_2'' - 2\gamma\mu_1' + \Gamma\gamma^2\mu_2 + \Gamma\gamma\mu_1' + \mu_4 - \ddot{\mu}_2 - \gamma^2\mu_2 = 0, \tag{18e}$$
$$\Gamma\mu_3'' - 2\ddot{\mu}_3 = 0. \tag{18f}$$

This system has a set of fundamental solutions which we label by the spatial mode number n:

$$\mu_n^{(1)} = \xi_n \mathrm{e}^{\sigma t + \mathrm{i}ns}, \tag{19}$$

where $\xi_n \in \mathbb{C}^6$.

The growth rate $\sigma = \sigma(n)$ is determined by the dispersion relation: $\Delta(\sigma, n) = 0$ obtained by substituting (19) into (16), namely

$$\Delta = (\gamma^2 - n^2)[(\gamma^2(\Gamma - 1) - P^2 - n^2)^2 - \gamma^2(\Gamma - 2)^2 n^2]. \tag{20}$$

Any solution $\mu^{(1)}$ is a linear superposition of the modes (19),

$$\mu^{(1)} = \sum_n X_n \mu_n^{(1)} + X_n^* (\mu_n^{(1)})^*, \tag{21}$$

where $(\ )^*$ stands for the complex conjugate and the X_n's are arbitrary amplitudes.

Of particular interest for the stability analysis of the stationary solutions are the *neutral modes* which are the modes associated with $\sigma = 0$ (see Fig. 1). Depending on the parameters, two such modes can be found. The first one is present for all values of the parameters and is associated with the mode $n = 0$. It corresponds to an arbitrary rotation around the central axis. The second neutral mode is a *helix* which is obtained when

$$n_c = \frac{P(2 - \Gamma)}{\Gamma}, \qquad \gamma_c = \pm 2\frac{P}{\Gamma}. \tag{22}$$

This defines a neutral curve in parameter space which identifies the region between stable and unstable solutions. For fixed $P > 0$ and $\gamma > \gamma_c$, the straight rod is unstable. The solution $n = \gamma$ of the dispersion relation is not an actual solution of the stationary equations since for this special value of the parameter all orders of perturbation cancel identically (i.e., $\alpha^{(i)} = 0\ \forall i$).

Before proceeding, we introduce the linear operator adjoint to (16);

$$L_E^{\dagger}(f^{(0)}, \kappa^{(0)}) \cdot \nu^{(1)} = 0, \tag{23}$$

whose neutral modes are

$$\nu_n^{(1)} = \zeta_n \mathrm{e}^{\mathrm{i}ns}, \tag{24}$$

where n is again determined by $\Delta(0, n) = 0$.

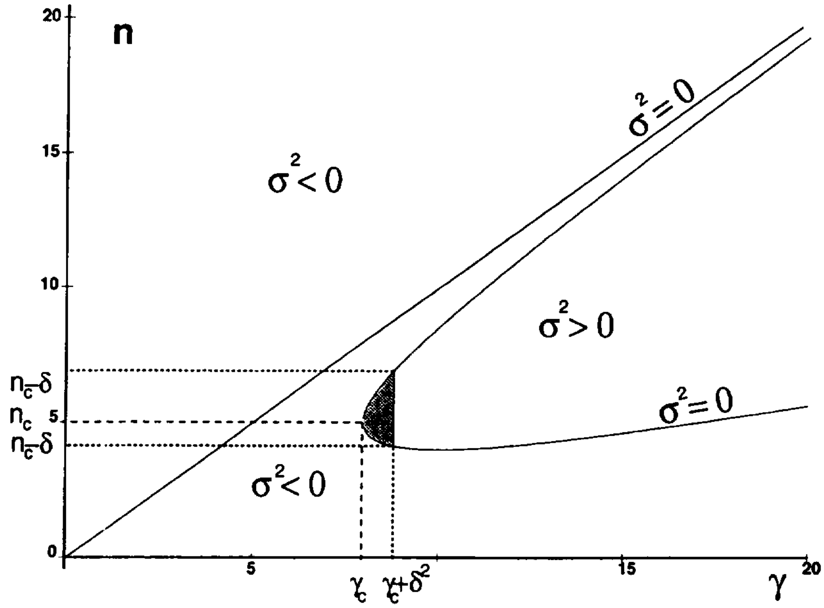

Fig. 1. A typical plot of the dispersion relation ($P = 3$). The straight line is the null solution while the parabola is the neutral curve defining the helix. The dashed zone is the region where unstable modes can be excited.

The linear analysis identifies the initial instabilities as a function of the parameters and the growth rate of the new solutions. However, the analysis is limited in many aspects. First, it is only valid for short times. Indeed, the exponential (in t) divergence of the (linear) solution leads to the break down of the assumption that these solutions are of order $O(\epsilon)$. As the linear solution grows, the nonlinear terms, neglected in the linear approximation, cannot be ignored. Therefore, the linear analysis is only valid as an indication of the parameter values at which the stationary solutions become unstable. For longer times, the nonlinear terms have to be taken into account. A second limitation of the linear analysis comes from the choice of initial conditions. The linear solutions that we obtained are only valid for infinitely long rods (or equivalently, when n is an integer, for periodic boundary conditions). For a finite rod with given boundary conditions, there is no linear solution. The nonlinear analysis allows us to overcome all these limitations.

4.2. Amplitude equations

The main idea behind an amplitude expansion is to consider the system close to bifurcation. In this regime, the distance from the bifurcation point is of the order of the perturbation itself. This relationship can then be used to introduce new, longer, space and time scales on which the arbitrary constants can vary. Here we consider the twist, γ, as the control (or "stress") parameter. A similar analysis can also be performed by taking tension as the control parameter. Thus, we set

$$\delta^2 = \gamma - \gamma_c. \tag{25}$$

For δ small enough, we can set $\delta = \epsilon^p$. The relationship between the two scales δ and ϵ can be found by expanding the dispersion relation in terms of δ,

$$\Delta(\sigma, n) = \Delta_0(0, n_0) + \delta\Delta(\sigma_1, n_1) + O(\delta^2), \tag{26}$$

where we have set $\sigma = \delta\sigma_1 + O(\delta^2)$ and $n = n_0 + \epsilon n_1 + O(\delta^2)$.

Introducing the stretched time and space scales,

$$t_0 = t, \qquad t_1 = \epsilon t, \tag{27}$$

$$s_0 = s, \qquad s_1 = \epsilon s, \tag{28}$$

we can conclude from an analysis of the dispersion relation that $\delta = \epsilon$.

Now, taking into account the expansion in the bifurcation parameter and the new scales, we look for solutions of the full system order by order in ϵ:

$$O(\epsilon^0): \quad E(\mu^{(0)}; s_0, t_0) = 0, \tag{29a}$$

$$O(\epsilon^1): \quad L_E(\mu^{(0)}; s_0, t_0) \cdot \mu^{(1)} = 0, \tag{29b}$$

$$O(\epsilon^2): \quad L_E(\mu^{(0)}; s_0, t_0) \cdot \mu^{(2)} = H_2(\mu^{(1)}), \tag{29c}$$

$$O(\epsilon^3): \quad L_E(\mu^{(0)}; s_0, t_0) \cdot \mu^{(3)} = H_3(\mu^{(1)}, \mu^{(2)}), \tag{29d}$$

$$\vdots$$

where H_i is polynomial in its arguments and derivatives.

To order $O(\epsilon)$, the (linear) solution is given by a superposition of the neutral modes, namely

$$\mu^{(1)} = X_0(s_1, t_1)\xi_0 + X_n(s_1, t_1)\xi_n e^{ins_0} + X_n^*(s_1, t_1)\xi_n^* e^{-ins_0}, \tag{30}$$

where $X_0(s_1, t_1)$ and $X_n(s_1, t_1)$ represent, respectively, the slowly varying amplitudes of the axial twist and the unstable helical mode; $n = n_c$; and

$$\xi_0 = (0, 0, 1, 0, 0, 1), \qquad \xi_n = (1, i, 0, -iP^2, P^2, 0). \tag{31}$$

At this order of ϵ the functions X_0 and X_1 are arbitrary and are constant on the scales (s_0, t_0) but may vary on the longer scales (s_1, t_1).

In the representation of the linear solution $\mu^{(1)}$ we have omitted the contribution of the oscillating modes, namely the solutions (19) with $\sigma^2 < 0$. These modes oscillate in time and are not unstable. This is a direct consequence of the conservative character of the system. In fluid mechanical contexts where amplitude equations are usually derived, the dissipative character of the physical system implies that most of the modes are damped exponentially fast in time and can therefore be neglected in the analysis. By contrast, no similar situation holds here and there is a possibility that these modes are relevant for the dynamics. However, we show in the next section that the amplitude equations obtained capture most of the relevant physical characteristics of the system and we, therefore, conjecture that these oscillatory modes can be omitted in the nonlinear analysis. This is similar to the considerations used in the study of the beam equation performed by Lange and Newell [20] (see Section 5).

We now derive an amplitude equation describing the slow evolution of the rod in the time scale s_1, t_1. In order to do so, we consider the higher-order equations in (29) and look for conditions on the amplitude to ensure that the solution to remains bounded. To order $O(\epsilon^2)$ no such condition appears and the system can be solved for $\mu^{(2)}$:

$$\mu^{(2)} = \chi_n e^{ins_0} + \chi_n^* e^{-ins_0} \tag{32}$$

with

$$\chi_n = (0, 0, 0, \tfrac{1}{2}X_n X_0 P^2, \tfrac{1}{2}iX_n X_0 P^2, 0). \tag{33}$$

We only consider a particular solution of the linear equation (29c) since the general solution is already included in the first-order solution. Rather than finding the solution to third-order we now derive conditions on the amplitudes X_0, X_n for the first-order solution to be bounded in space. To do so, we apply the Fredholm alternative to the system (29d). Here this consists of integrating H_3 against all neutral solutions (24) of the adjoint operator $L_E^\dagger$:

$$\int_0^{2\pi/n} \zeta_0 \cdot H_3(\mu^{(1)}, \mu^{(2)})\, \mathrm{d}s_0 = 0, \tag{34a}$$

$$\int_0^{2\pi/n} \mathrm{e}^{\mathrm{i}ns_0}\zeta_n \cdot H_3(\mu^{(1)}, \mu^{(2)})\, \mathrm{d}s_0 = 0, \tag{34b}$$

$$\int_0^{2\pi/n} \mathrm{e}^{-\mathrm{i}ns_0}\zeta_n^* \cdot H_3(\mu^{(1)}, \mu^{(2)})\, \mathrm{d}s_0 = 0, \tag{34c}$$

where the product $a \cdot b$ is the standard scalar product and H_3 is given by:

$$H_{31} = -\mathrm{i}\mathrm{e}^{\mathrm{i}ns_0}\left(\frac{\partial^2 X_n}{\partial t_1^2} + 2P^4 X_n|X_n|^2\right) + \mathrm{i}\mathrm{e}^{-\mathrm{i}ns_0}\left(\frac{\partial^2 X_n^*}{\partial t_1^2} + 2P^4 X_n^*|X_n|^2\right), \tag{35a}$$

$$H_{32} = \mathrm{e}^{\mathrm{i}ns_0}\left(\frac{\partial^2 X_n}{\partial t_1^2} - 2P^4 X_n|X_n|^2\right) + \mathrm{e}^{-\mathrm{i}ns_0}\left(\frac{\partial^2 X_n^*}{\partial t_1^2} + 2P^4 X_n^*|X_n|^2\right), \tag{35b}$$

$$H_{33} = 0, \tag{35c}$$

$$\begin{aligned} H_{34} = {} & \mathrm{e}^{\mathrm{i}ns_0}\left(-\frac{\partial^2 X_n}{\partial t_1^2} + \frac{\partial^2 X_n}{\partial s_1^2} + P\Gamma X_n\frac{\partial X_0}{\partial s_1} + P\Gamma X_n - 2P^2(\Gamma - 1)X_n|X_n|^2\right) \\ & + \mathrm{e}^{-\mathrm{i}ns_0}\left(-\frac{\partial^2 X_n^*}{\partial t_1^2} + \frac{\partial^2 X_n^*}{\partial s_1^2} + P\Gamma X_n^*\frac{\partial X_0}{\partial s_1} + P\Gamma X_n^* - 2P^2(\Gamma - 1)X_n^*|X_n|^2\right), \end{aligned} \tag{35d}$$

$$\begin{aligned} H_{35} = {} & \mathrm{i}\mathrm{e}^{\mathrm{i}ns_0}\left(-\frac{\partial^2 X_n}{\partial t_1^2} + \frac{\partial^2 X_n}{\partial s_1^2} + P\Gamma X_n\frac{\partial X_0}{\partial s_1} + P\Gamma X_n + 2P^2(\Gamma - 1)X_n|X_n|^2\right) \\ & - \mathrm{i}\mathrm{e}^{-\mathrm{i}ns_0}\left(-\frac{\partial^2 X_n^*}{\partial t_1^2} + \frac{\partial^2 X_n^*}{\partial s_1^2} + P\Gamma X_n^*\frac{\partial X_0}{\partial s_1} + P\Gamma X_n^* + 2P^2(\Gamma - 1)X_n^*|X_n|^2\right), \end{aligned} \tag{35e}$$

$$H_{36} = -2P\Gamma\frac{\partial|X_n|^2}{\partial s_1} + \Gamma\frac{\partial^2 X_0}{\partial s_1^2} - 2\frac{\partial^2 X_0}{\partial t_1^2}. \tag{35f}$$

Taking into account the specific form of $\mu^{(1)}$ and $\mu^{(2)}$ given by (30)–(32), we obtain two equations for the amplitudes $X_0 \equiv Y, X_n \equiv X$:

$$\left(\frac{P^2+1}{P^2}\right)\frac{\partial^2 X}{\partial t_1^2} - \frac{\partial^2 X}{\partial s_1^2} = P\Gamma X\left(1 - 2P|X|^2 + \frac{\partial Y}{\partial s_1}\right), \tag{36a}$$

$$\frac{2}{\Gamma}\frac{\partial^2 Y}{\partial t_1^2} - \frac{\partial^2 Y}{\partial s_1^2} = -2P\frac{\partial|X|^2}{\partial s_1}. \tag{36b}$$

These equations are a system of two coupled nonlinear Klein–Gordon equations. They couple the local deformation of the rod X with the twist density Y. The twist density plays a central role in these equations. Indeed, if we set $Y = 0$, it is easy to see that the stationary solutions may blow up. Certain features of the amplitude equations can also be understood in terms of symmetry-breaking [28]. From this perspective, it can be seen that the first-order derivatives with respect to s_1 break the symmetry associated with the rotation of the rod about the central axis and as a result introduce a twist-imposed handedness in the post-bifurcation solution.

Physically, these equations can also be seen as the generalization of the torsional wave equation. Indeed, the torsional wave equation describes the evolution of a twist "wave" in an infinite rod propagating at speed $c^2 = \frac{1}{2}\Gamma$ (see for instance [23]). Neglecting the nonlinear terms in (36b), we obtain the torsional wave equation for the twist density Y. Therefore, these equations are the nonlinear version of the torsional wave equation where the deformation of the rod X is taken into account as the twist wave propagates along the rod.

4.3. Particular solutions of the amplitude equations

It is well-known that the nonlinear Klein–Gordon equations are non-integrable, in the sense that they do not exhibit any of the accepted characteristics of integrable evolution equations such as Lax pairs, soliton solutions, infinite conservations laws, bi-Hamiltonian formulation and so on. Indeed, an application of the WTC algorithm (see [29]) shows that our system fails the Painlevé test for partial differential equations which is indicative of non-integrability. Nonetheless, it is still possible to obtain certain particular solutions explicitly. Although, we have not studied the stability of these solutions they appear to be physically interesting for the problem under consideration.

4.3.1. Homogeneous solutions

We first drop the spatial dependence. The system reduces to

$$\frac{\partial^2 X}{\partial t_1^2} = \frac{P^3 \Gamma}{P^2+1} X(1 - 2P|X|^2), \tag{37a}$$

$$\frac{\partial^2 Y}{\partial t_1^2} = 0 \tag{37b}$$

and the twist density decouples from the deformation. Therefore we set $Y(t_1) = K$, where K is an arbitrary constant (the twist density cannot increase indefinitely in time). Without loss of generality (that is up to an arbitrary rotation around the rod axis), we can consider real solutions for which $X = X^*$ and the amplitude of the rod deformation follows the time evolution of a Hamiltonian system with a quartic potential:

$$\frac{\partial^2 X}{\partial t_1^2} = -\frac{\partial}{\partial X} V(X), \tag{38a}$$

$$V = -\frac{P^3 \Gamma}{2(P^2+1)} X^2(1 - PX^2). \tag{38b}$$

After integration of (13), the filament solution is found to correspond to a helix

$$x(s,t) = \left(s, \frac{-2\epsilon X(\epsilon t)}{P} \sin Ps, \frac{-2\epsilon X(\epsilon t)}{P} \cos Ps\right). \tag{39}$$

A typical plot of the potential is shown in Fig. 2, together with the shape of the rod for different times and the evolution of the amplitude as a function of time. Starting from a small perturbation of the infinite rod, the rod oscillates periodically in time between the straight configuration and the helical shape (configurations 2a and 2d). In the presence of small viscosity, it is expected that the rod will eventually reach the minimum of the potential

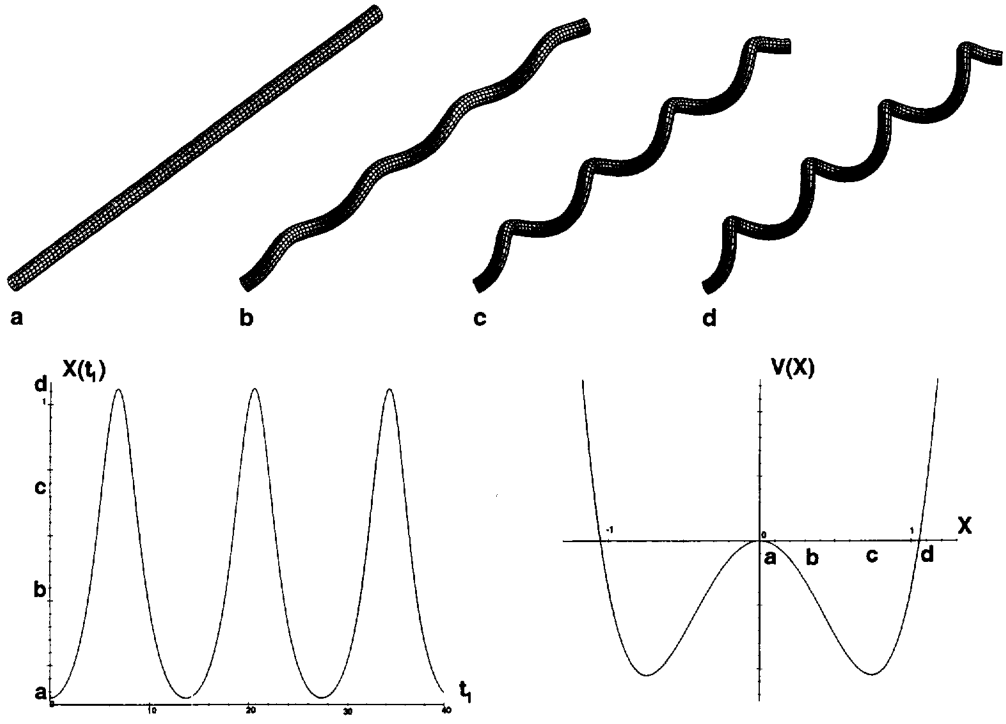

Fig. 2. The homogeneous solutions: a typical plot of the potential for ($P = \frac{9}{10}$, $\Gamma = \frac{3}{4}$) together with the different rod configurations and the time evolution of the amplitude for the initial condition $X(0) = 0.1$.

(configuration 2c). However, the whole dynamical instability may be modified when the viscosity becomes large and, eventually, a new nonlinear analysis should be performed for these cases.

4.3.2. Traveling wave solutions

Setting $z = s_1 - ct_1$ one obtains the traveling wave reduction of the amplitude equations, i.e.

$$\frac{\partial^2 X}{\partial z^2} = \frac{P^3\Gamma}{P^2(c^2-1)+c^2} X\left(1 - 2P|X|^2 + \frac{\partial Y}{\partial z}\right), \tag{40a}$$

$$\left(\frac{2c^2-\Gamma}{\Gamma}\right)\frac{\partial^2 Y}{\partial z^2} = -2P\frac{\partial |X|^2}{\partial z}. \tag{40b}$$

The second equation can be integrated once and used to simplify the first one, namely

$$\frac{\partial^2 X}{\partial z^2} = \frac{P^3\Gamma}{P^2(c^2-1)+c^2} X\left(1 + K + \frac{2c^2 - P\Gamma}{2c^2-\Gamma}|X|^2\right), \tag{41}$$

where K is an arbitrary constant chosen in such a way that the derivative of the twist goes to zero at infinity.

For real solutions ($X = X^*$), this equation can again be put in a form corresponding to motion in a quartic potential,

$$V(X) = -\frac{P^3\Gamma X^2}{(P^2c^2 - P^2 + c^2)}\left(\frac{Pc^2X^2}{(\Gamma - 2c^2)} + \frac{1+K}{2}\right), \tag{42}$$

where we have chosen solution with zero twist at infinity, that is $K = 0$.

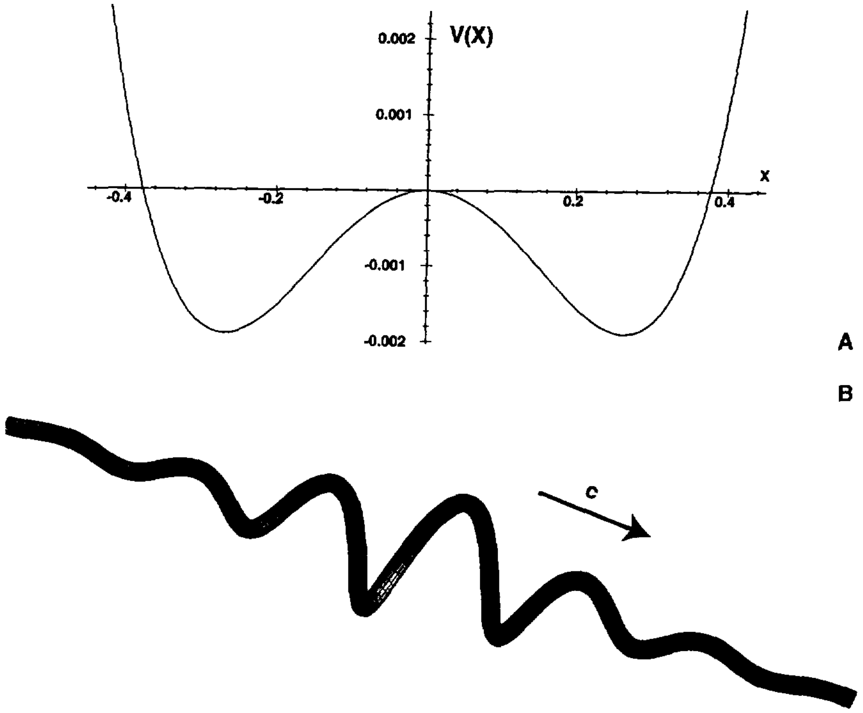

Fig. 3. Traveling waves solutions: The potential for $c = 7$ and ($P = 7, L = 1, \Gamma = \frac{3}{4}$) together with the pulse-like solution corresponding to the homoclinic orbit connecting the maximum of the potential to itself.

For $c^2 > \frac{1}{2}\Gamma$, two interesting situations arise. First if $c^2 > \max\{\frac{1}{2}\Gamma, P^2/(P^2+1)\}$ the potential (shown in Fig. 3A) sustains (among others solutions), homoclinic orbits of the form

$$X(z) = \rho_1 \operatorname{sech}(\rho_2 z) \tag{43}$$

with $\rho_1^2 = (2c^2 - \Gamma)/2c^2 P$, $\rho_2^2 = P^3\Gamma/(P^2c^2 - P^2 + c^2)$, $K = 0$.

The corresponding filament is obtained by (13) and is shown in Fig. 3(B). This *pulse-like* solitary wave solution travels along the rod with constant speed c. In a similar way, the periodic solutions inside the potential well-correspond to trains of pulse-like solutions periodically distributed along the axis and moving at the same constant speed.

Another type of solution can be obtained for $P^2/(P^2+1) > c^2 > \frac{1}{2}\Gamma$. For these values of c the potential (shown on Fig. 4(B)) exhibits heteroclinic connections between the two potential maxima, namely

$$X(z) = \rho_1 \tanh(\rho_2 z) \tag{44}$$

with $\rho_1^2 = 1/2P$, $\rho_2^2 = P^3\Gamma/[(\Gamma - 2c^2)(P^2c^2 - P^2 + c^2)]$, $K = \Gamma/(\Gamma - 2c^2)$.

These heteroclinic connections describe *front-like* solitary waves connecting two different asymptotic states (see Fig. 4(B).)

We comment that an explicit closed form of the solution $x = x(s, t)$ cannot be obtained in general.

We conclude that there is, for given physical parameters, a one-parameter (depending on c) family of traveling waves solutions. Can they be observed and if so, is there a selection mechanism for the different speeds? In recent years many traveling wave problems have been studied successfully in the theory of reaction–diffusion equations [30] but, as of now, the stability of the solutions to the problem presented here remains an interesting open question.

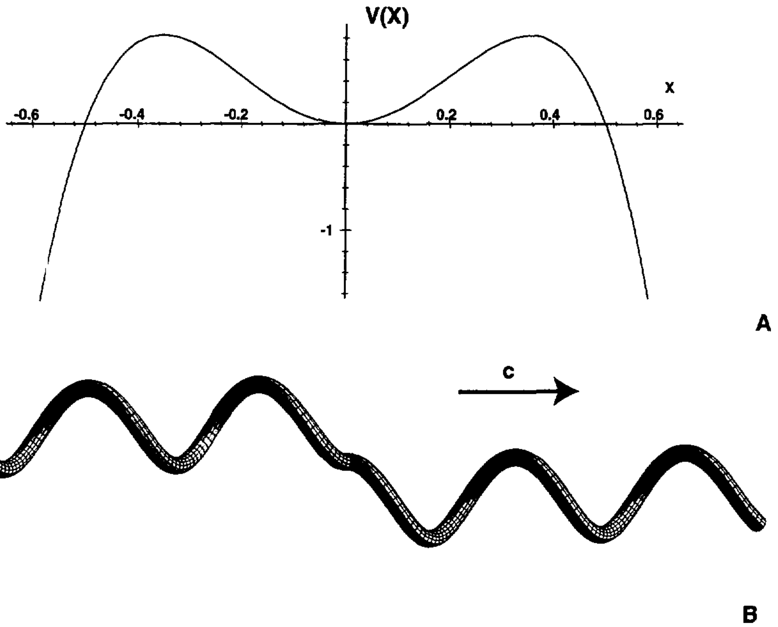

Fig. 4. Traveling waves solutions. The potential for $c = \frac{7}{10}$ and ($P = 4, L = 1, \Gamma = \frac{3}{4}$) together with the front-like solution corresponding to the heteroclinic orbit connecting the two potential maxima.

There has been some recent work concerning the existence of traveling waves solutions for the two- and three-dimensional Kirchhoff [27,31–33] equation. These so-called *flexural waves* are exact solution of the Kirchhoff equation and are obtained directly from Eqs. (4). However, to date, these solutions do not seem related to the (approximate) solutions obtained here. Indeed these flexural waves are obtained with a constant twist condition whereas the solutions presented here have non-constant twist. It would be of interest to see if our solutions can be obtained as an approximation of exact traveling waves solutions.

4.3.3. Stationary solutions

The study of stationary solutions of the Kirchhoff equations has a long tradition in mechanics and engineering [23]. There are many different results for the straight rod with a variety of boundary conditions. We show that the amplitude equations can also be used as a general equation to find different stationary solutions for different types of boundary conditions.

The stationary amplitude equations read:

$$\frac{\partial^2 X}{\partial s_1^2} = -P\Gamma X\left(1 - 2P|X|^2 + \frac{\partial Y}{\partial s_1}\right), \tag{45a}$$

$$\frac{\partial^2 Y}{\partial s_1^2} = 2P\frac{\partial |X|^2}{\partial s_1}. \tag{45b}$$

As it turns out, the coupling between the variables X and Y in the amplitude equations is such that the equation for X is linear, the second equation can be readily integrated, i.e. $\partial Y/\partial s_1 = 2P|X|^2 - 1 - K_1$, and the result substituted into the first equation to yield

$$\frac{\partial^2 X}{\partial s_1^2} = P\Gamma X K_1. \tag{46}$$

Hence the stationary solution is

$$X(s_1) = K_2 e^{i\sqrt{P\Gamma K_1}s_1} + K_3 e^{-i\sqrt{P\Gamma K_1}s_1}, \tag{47a}$$

$$Y(s_1) = \sqrt{\frac{PK_1}{\Gamma}}\left(K_2K_3^* e^{2i\sqrt{P\Gamma K_1}s_1} - K_2^*K_3 e^{-2i\sqrt{P\Gamma K_1}s_1}\right) + s_1\left(-1 - K_1 + 2P(|K_2|^2 + |K_3|^2)\right) + K_4. \tag{47b}$$

We now consider a finite straight rod of length $2\pi L$. The extremities are held fixed, hence they cannot be perturbed away. Therefore, we have as boundary conditions: $X(0) = X(2\pi L\epsilon) = Y(0) = Y(2\pi L\epsilon) = 0$. As a consequence, the arbitrary constants K_i are:

$$K_1 = \frac{L^2}{4P\Gamma\epsilon^2}, \tag{48a}$$

$$|K_2|^2 = \frac{1}{4}\left(1 - \frac{\pi}{L^2\epsilon^2P^2\Gamma}\right), \tag{48b}$$

$$|K_3|^2 = \frac{1}{4}\left(1 - \frac{\pi}{L^2\epsilon^2P^2\Gamma}\right), \tag{48c}$$

$$K_4 = 0. \tag{48d}$$

Note that K_2 and K_3 are defined up to an arbitrary phase corresponding to an arbitrary rotation around the rod axis. To remove this arbitrariness, we choose $K_2 = -K_2^*$ and $K_3 = -K_3^*$. The solution for the rod is, after integration of (13):

$$x(s) = s, \tag{49a}$$

$$y(s) = \frac{\sqrt{4L^2\epsilon^2P\Gamma - 1}}{2PL\sqrt{\Gamma}a_+a_-}\left(a_-\cos a_+s - a_+\cos a_-s\right), \tag{49b}$$

$$z(s) = \frac{\sqrt{4L^2\epsilon^2P\Gamma - 1}}{2PL\sqrt{\Gamma}a_+a_-}\left(a_-\sin a_+s - a_+\sin a_-s\right), \tag{49c}$$

where $a_\pm = P \pm 1/2L$. We note that the amplitude of these solutions is real only for $\epsilon^2 \geq 1/4L^2P\Gamma$. That is, the bifurcation does not occur for $\gamma = \gamma_c$ but for $\gamma = \gamma_c + (1/4P\Gamma L^2)$. This is due to the boundary conditions. Indeed, by constraining the rod to be straight at the ends, the bifurcation is delayed. The delay varies as L^{-1}. Therefore, as $L \to \infty$ one recovers the bifurcation value for the infinite rod. The solutions (49a) are shown in Fig. 5 for some typical values of the parameters. This delay phenomenon has been experimentally observed in several similar physical settings [34,35] and has also been considered in fluid dynamics [36,37].

5. The beam equations

We have studied the instability of straight filaments after bifurcation in the framework of the Kirchhoff model. However, the usual approach to understanding the instability of straight filaments has been to use the so-called *beam equation* [24] and a considerable amount of theoretical work has been performed on this equation. We focus here on the application of nonlinear analysis to this equation and compare it to the results obtained in the previous sections. The beam equation describes the space and time evolution of small displacement $W(s,t)$ of a straight filament under axial load:

$$\rho\frac{\partial^2 W}{\partial t^2} + EI\frac{\partial^4 W}{\partial s^4} + P\frac{\partial^2 W}{\partial s^2} + \xi_1 W + \xi_2 W^m = 0, \tag{50}$$

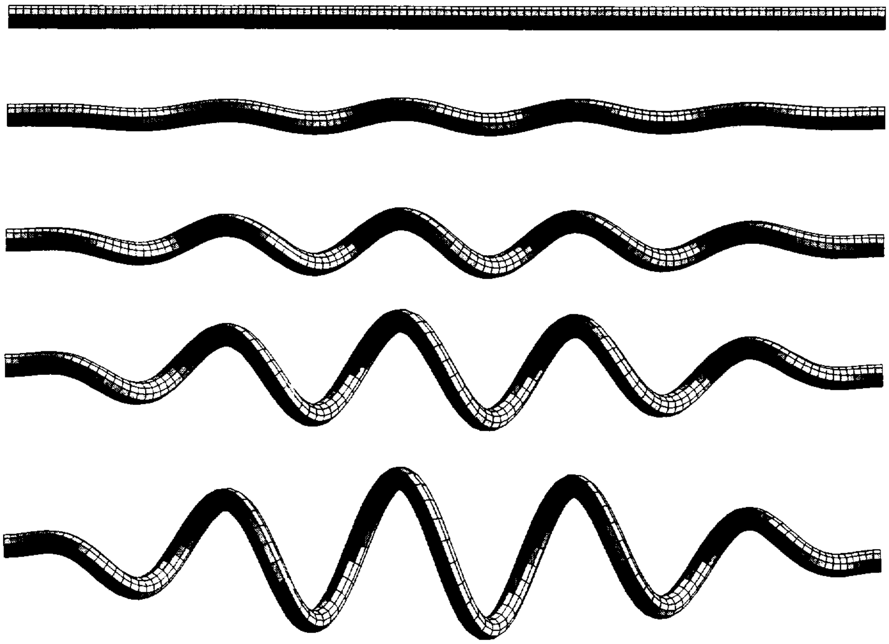

Fig. 5. The stationary solutions: Different stationary solutions for ($P = \frac{9}{10}$, $L = 1$, $\Gamma = \frac{3}{4}$) and for increasing values of ϵ. Note that due to the boundary conditions the straight rod is stable up to $\epsilon = 0.29$.

where ρ is the mass per unit length, EI, the bending stiffness and P the axial load.

The first problem comes with the choice of the exponent m and the coefficients ξ. Different authors choose different types of nonlinear response and the relationship with given experimental data or microscopic quantities seems hard to achieve. A common choice for the exponent is $m = 3$ [19,20], whereas $m = 2$ has also been used [38]. In the same way, the sign of ξ_2 is not determined. It is clear that a different choice of exponents or different choices of sign will lead to radical changes in the dynamical behavior after bifurcation. Let us consider the most widely studied case $m = 3$ and $\xi_2 < 0$. After rescaling, the beam equation can be put in the form

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial s^4} + 2\lambda \frac{\partial^2 w}{\partial s^2} + w - w^3 = 0. \tag{51}$$

A linear and nonlinear analysis similar to the one we presented here can be easily performed for this equation and an amplitude equation for the mode $w(s,t) = A(s_1, t_1)e^{ins_0}$ + c.c is obtained:

$$\frac{\partial^2 A}{\partial t_1^2} - 4\frac{\partial^2 A}{\partial s_1^2} = A(2\chi + 2|A|^2), \tag{52}$$

where $\lambda = 1 + \epsilon\chi$, $\chi > 0$ gives the distance from the bifurcation point.

Eq. (52) is a nonlinear Klein–Gordon equation. The homogeneous solutions $A = A(t_1)$ are all unbounded, the amplitude of the rod blows up in a finite-time and the unstable linear modes are never saturated. The stationary solutions, on the other hand can accommodate the same boundary conditions as studied here. Overall, we see that a consistent dynamical analysis cannot be carried out on the beam equations since the slow time dependence on the linear mode is not bounded. Different variations of the beam equations have been proposed in order to resolve this problem [21,22,38–41]. However, none of these studies were successful in providing an amplitude equations for the solutions after bifurcation which could satisfactorily explain both space and time dependence. Most of them limited their scope to time-independent equations and gave a *qualitative* picture of the possible phenomena encountered in experimental settings [34,35,42].

From the analysis we have performed on the full Kirchhoff model, we see that the coupling of the displacement to the twist density – which in turn requires a full three-dimensional description – is crucial in explaining the dynamical instability of the solutions. It is only due to this coupling that the linear modes can saturate in time.

6. Conclusions

The problem of the stability of stationary solutions in elasticity theory has a long history. However, to the best of our knowledge no results concerning the *dynamical character* of these instabilities have been available. In order to understand this question, we investigated the problem in the general setting of the Kirchhoff model. In [15], we developed a new perturbation scheme for these equations in the director frame and give a complete picture of the linear instability of stationary solutions. In the present work, we completed this study by carrying out a nonlinear analysis of the straight rod after bifurcation. Doing so, we found a new nonlinear evolution equation, a system of two nonlinear Klein–Gordon equations coupling the rod displacement with the twist density. We investigated some physically important particular solutions of these amplitude equations and showed the existence of intriguing pulse-like and front-like solitary waves. In addition, the effect of boundary conditions for stationary solutions was explored and shown to delay the bifurcation. Many other studies can be carried out on these nonlinear equations such as extensive numerical investigation; stability of particular solutions; secondary bifurcations and so on. A similar type of nonlinear analysis can be performed on different filamentary structures such as the closed ring or the helix [43] which are of importance in many biological problems.

The complex Ginzburg–Landau plays a key-role in nonlinear science as a "normal form" describing the spatio-temporal structure of supercritical bifurcations in many dissipative systems. We believe that the nonlinear Klein–Gordon system we derived here could play the same role for conservative systems.

Acknowledgements

This work is supported by DOE grant DE-FG03-93-ER25174. The authors would like to thank J. Lega for many interesting discussions.

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