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Finite-time blow-up in dynamical systems

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Abstract

A new method to detect finite-time blow-up in systems of ordinary differential equations is presented. This simple algorithmic procedure is based on the analysis of singularities in complex time and amounts to checking the real-valuedness of the leading order term in the asymptotic series describing the behavior of the general solution around movable singularities. Illustrative examples and an application to a magneto-hydrodynamic problem are given. © 1998 Elsevier Science B.V.

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1. Introduction

The problem of finite-time blow-up for partial differential equations (PDEs) is a most active domain of research in applied mathematics [1-3]. Literally hundreds of papers have been written on the problem for different sets of equations, and fundamental physical problems, such as the existence of solutions for the Euler equations, rely on the analysis of finite-time singularities for PDEs [4,5]. However, despite the overwhelming interest of the applied mathematics community in this problem, the analog problem for ordinary differential equations (ODEs) has hardly been investigated [6]. To the best of our knowledge, the simple question of whether or not a system of ordinary differential equations exhibits finite-time blow-up has not been answered, or even thoroughly addressed. Here, *time* is understood as the independent variable for a system of ODEs and we consider systems of autonomous nonlinear polynomial ODEs,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{1}$$

where $\dot{\boldsymbol{x}} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ and boldface denotes vector. The *general solution* is a solution that contains n arbitrary constants and will be denoted $\boldsymbol{x} = \boldsymbol{x}(t; c_1, \dots, c_n)$. In the same way, the solution based on the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ will be $\boldsymbol{x} = \boldsymbol{x}(t; \boldsymbol{x}_0)$. A solution will exhibit *finite-time blow-up* if there exists $t_* \in \mathbb{R}$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that for all $M \in \mathbb{R}$, there exists an $\varepsilon > 0$ satisfying

$$|t - t_*| < \varepsilon \Rightarrow \| \boldsymbol{x}(t; \boldsymbol{x}_0) \| > M, \tag{2}$$

where $\| \cdot \|$ is any l^p norm. Equivalently, we use "$\lim_{t \to t_*} \| \boldsymbol{x}(t, \boldsymbol{x}_0) \| \to \infty$" to denote such a blow-up.

It is known that nonlinear ODEs exhibit *movable singularities* depending on the initial conditions. These singularities are complex valued functions of

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the initial conditions and therefore might not appear in dynamics constrained to the real axis. The problem of finding the location of the singularities as a function of the initial conditions cannot be solved in general. However, we show here that it is possible to find necessary and sufficient conditions for the existence of an open set of initial conditions leading to finite-time blow-up. That is, the solution based on this set of initial conditions will cease to exist after a finite time t_*.

In order to illustrate the problem and the method, we consider a one-degree of freedom Hamiltonian system with polynomial potential

$$\ddot{x} = ax^n + g(x), \tag{3}$$

where $g(x)$ is a polynomial of degree less than n with $g(0) = 0$, $n \geqslant 3$ and $a \neq 0$. This system has a Hamiltonian $H = \frac{1}{2}\dot{x}^2 + V(x)$ with potential $V(x) = -ax^{n+1}/(n+1) - \int^x g(x)\,dx$. Depending on the parity of n and the sign of a, this system can exhibit finite-time blow-up. Already it can be seen that not all trajectories diverge to infinity. Indeed, the fixed point $x = 0$ is a particular solution which does not exhibit finite-time blow-up. This is why we are interested in proving the existence of an open set of initial conditions rather than proving that all initial conditions lead to a blow-up. The analysis of the singularities of this system is straightforward when one considers the graph of the potential functions. Depending on the parity of n and the sign of a, four different cases can be discussed. (See Fig. 1.) If n is odd and a is negative (Fig. 1a), then all orbits are bounded in phase space and there is no possibility of blow-up. If n is odd and a is positive, then by choosing $|x|$ large enough ($|x| > x_c$ on Fig. 1b), an open set of initial conditions $\{x_0, \dot{x}_0\}$ leading to finite-time blow-up can be easily found. Moreover, for these initial conditions, the blow-up time t_* can be explicitly computed,

$$t_* = \int_{x_0}^{\infty} \frac{dx}{\sqrt{2[E - V(x)]}}, \tag{4}$$

with $E = H(x_0, \dot{x}_0)$. If the potential is uneven (n even), then independently of the choice of a, there always exists a critical value x_c such that $x > x_c$ (for $a > 0$) or $x < x_c$ (for $a < 0$) leads to a blow-up. However, the blow-up now occurs in only one quadrant of the phase space. The blow-up time can be again obtained by considering (4) wherever it applies. Finally, we observe that the lower-order terms $g(x)$ do not change the main result. Blow-up can be delayed but it cannot be avoided in the entire phase space. One of the possible effects of the lower terms $g(x)$ is to create regions of phase space where the solution is bounded (for instance the choice $\dot{x}_0 = 0$ and $|x_0| < x_1$ for the potential on Fig. 1b leads to periodic orbits or fixed points).

Now, we can compare this analysis with the analysis performed locally around the singularities. Around a singularity $t_* \in \mathbb{C}$, the following asymptotic expansions can be found,

$$x = \alpha(t_* - t)^p\,(1 + h(t_* - t)), \tag{5}$$

where $h(t_* - t)$ is, in general, a Taylor series in a root of its argument. Its explicit form can be found but is not relevant here. The leading term $\alpha(t_* - t)^p$ is related to a and n in the following way,

$$p = \frac{2}{1-n}, \quad \alpha^{(n-1)} = \frac{2(1+n)}{a(1-n)^2}. \tag{6}$$

We see that the asymptotic form of the solutions around the singularities depends only on the dominant term ax^n and not on the lower order terms. Depending on the sign of a and the parity of n, the leading coefficient α can be real or complex. If n is even, there always exists a root[2] $\alpha = \sqrt[n-1]{2(1+n)/a(1-n)^2} \in \mathbb{R}$. If a is positive and n is odd, there are two such real roots: $\alpha = \pm\sqrt[n-1]{2(1+n)/a(1-n)^2} \in \mathbb{R}$. However, if a is negative and n odd, there is no real root for α. These observations indicate that *whenever one of the leading coefficients of the asymptotic series is real, finite-time blow-up occurs.* Moreover, when blow-up occurs in two different quadrants of phase space (Fig. 1b), two different series with real leading coefficients can be found.

This simple example seems to indicate that there is a simple connection between the real-valuedness of the leading coefficient and the occurrence of blow-up. This connection was shown to remain valid in general for systems of ODEs [7]. It is the purpose of this paper to illustrate this result and show how it can be applied

[2] The root $a = \sqrt[b]{c}$ for $c > 0$ and $b \in \mathbb{R}$, is the positive real number a such that $a^b = c$.

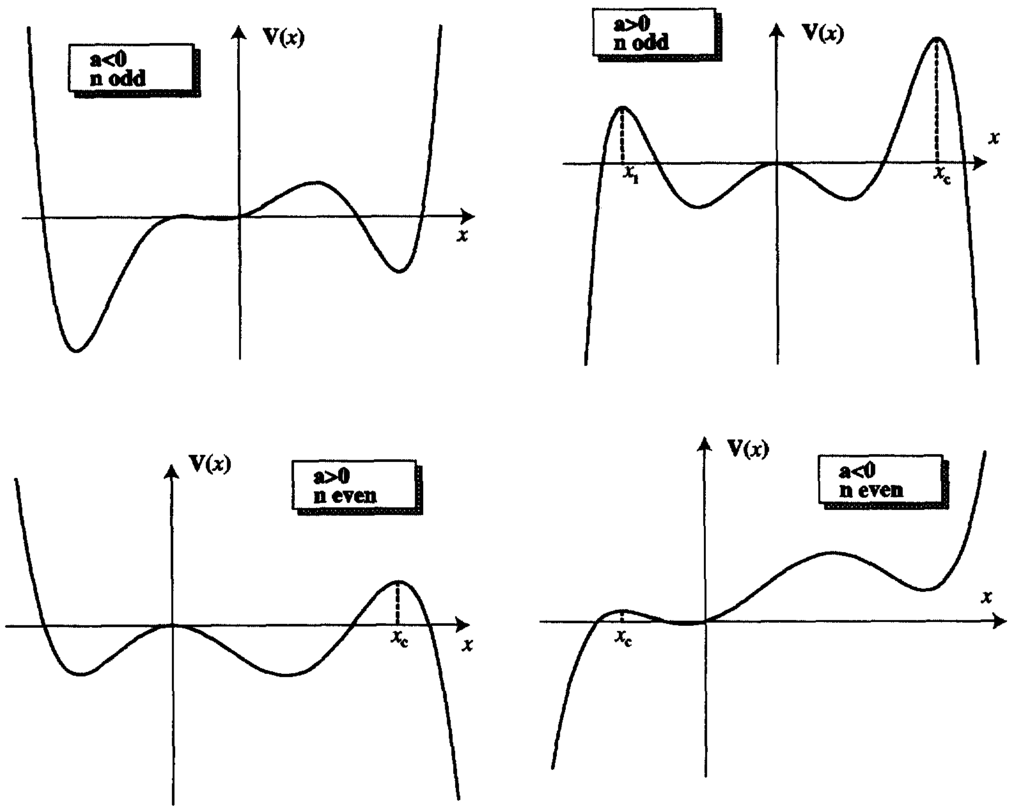

Fig. 1. The different possible potential configurations.

to particular dynamical systems arising in nonlinear science.

Before proceeding, we mention a few problems appearing in applied mathematics where the occurrence (or absence) of finite-time blow-up is relevant.

First, as mentioned earlier, in dynamical systems theory the first step of the analysis is to find the domain of existence of the solutions. If the solutions are defined for all time, the system is *complete* [8]. In the case where the solutions are only defined on a semi-interval, the system is *complete in positive (negative) time*. The usual way to prove the completeness of a system is by exhibiting a Lyapunov function, or a first integral, and showing that it controls the asymptotic behavior of the solutions. It is only recently that completeness for the Lorenz system was proved [8]. In general, a Lyapunov function cannot be found, and proving the domain of existence of the solution becomes an intricate, if not impossible, exercise.

Second, for Hamiltonian systems, the accepted definition of integrability is the so-called *Liouville integrability* [9]. Let $H = H(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be a Hamiltonian function of a canonical set of variables (x_k, y_k). This system is *Liouville integrable* if (i) there exist $n-1$ independent constants of motions $(J_1, \ldots, J_{n-1})$ in involution with H (that is $\{H, J_k\} = 0$ for all k); *and*, as stressed by Flaschka in Ref. [10], (ii) the n different systems of Hamilton's equations derived by taking H and the $n-1$ constants of motion $(J_1, \ldots, J_{n-1})$ as Hamiltonians have solutions defined for all time. As a consequence, in order to prove that a system is Liouville integrable, one has to not only exhibit $n-1$ constants of motion in involution but also show that the corresponding Hamiltonian flows based on these constants of motion do not exhibit finite-time blow-up. In some cases, the particular form of the constants of motion allows one to prove the absence of singularities. However, an analysis of the constants of motion cannot always decide on the existence of the flows. The method we describe here gives an explicit way of checking the existence of these flows.

A third application is fluid dynamics. Many authors have conjectured that the Euler equations lead to a divergence in finite time. In order to test this hypothesis, simplified models have been derived showing the spontaneous formation of singularities [11–14]. In the same way, the formation of singularities in the ideal equations of incompressible magneto-hydrodynamics has been shown to have important physical implications such as the occurrence of solar flares in the solar dynamo problem. Simplified models reduce the equations of magneto-hydrodynamics to simple systems of ODEs where the existence of real time singularities has to be proved [16].

The fourth application concerns the existence of singularities for PDEs. As we mentioned earlier, the existence of singularities in PDEs is a major problem in applied mathematics. We do not claim that the results described here could be easily generalized to the case of PDEs. However, in many instances, the process of proving the occurrence of blow-up in the solutions of PDEs reduces to the analysis of systems of ODEs controlling the blow-up [4]. The analysis of these systems is, in many cases, a straightforward application of our results.

The fifth potential application of our method is the numerical computation of blow-up. There exist many different numerical methods for computing blow-up in differential equations [17]. The main problem is to be able to differentiate between a blow-up induced by a numerical scheme and the intrinsic blow-up of the equations themselves. The methods that we develop here may provide an alternative way to test whether a large class of systems exhibit blow-up and could be used to decide on the most appropriate numerical methods taking into account the occurrence or absence of blow-up.

2. Main theorem

In order to study the occurrence of blow-up, we analyze the solutions locally around their singularities. This is done in the framework of *singularity analysis*, a set of methods based on the construction of local series around movable singularities. It is typically used to prove the integrability of systems of ODEs through the so-called *Painlevé property* and *Painlevé test* [18,19] but has been shown more recently to provide valuable insight into the dynamics of nonintegrable systems [20]. In order to analyze the occurrence of blow-up for solutions of system (1), we build local series of the form [21]

$$\boldsymbol{x} = \boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{p}, t) \equiv \boldsymbol{\alpha}\tau^{\boldsymbol{p}} \left(1 + h(\tau, \log\tau)\right), \qquad (7)$$

where $\tau = t_* - t$ and $h(\tau^q, \log\tau)$ is a power series in its argument which vanishes as $\tau \to 0$. The notation $\boldsymbol{\alpha}\tau^{\boldsymbol{p}}$ refers to the vector whose ith component is $\alpha_i\tau^{p_i}$. In the rest of this paper, we restrict the class of systems under consideration so that they have series solutions (referred to hereafter as *ψ-series*) with $\boldsymbol{p}$ and q rational. That is the ψ-series is a *Puiseux series* in τ whose coefficients are polynomial in $\log\tau$. In this case, it has been repeatedly argued that $h(\cdot,\cdot)$ is a *convergent* power series in its arguments for τ small enough [23–32]. This fact will be of importance in the following.

In order to obtain the leading behavior $\boldsymbol{\alpha}\tau^{\boldsymbol{p}}$ of the solution around t_* we look for all the truncations $\hat{\boldsymbol{f}}$ of the vector field $\boldsymbol{f} = \hat{\boldsymbol{f}} + \check{\boldsymbol{f}}$ such that the *dominant behavior* $\boldsymbol{x} = \boldsymbol{\alpha}\tau^{\boldsymbol{p}}$, $\boldsymbol{\alpha} \in \mathbb{C}_0^n$ is an exact solution of the truncated system $\dot{\boldsymbol{x}} = \hat{\boldsymbol{f}}(\boldsymbol{x})$ and

$$\check{\boldsymbol{f}}(\boldsymbol{\alpha}(t_* - t)^{\boldsymbol{p}}) \underset{t \to t_*}{\sim} \check{\boldsymbol{\alpha}}(t_* - t)^{\boldsymbol{p}+\check{\boldsymbol{p}}-1}, \qquad (8)$$

with $\check{\boldsymbol{p}} \in \mathbb{Q}^n$ and each $\check{p}_i > 0$. Each truncation defines a *balance* $(\boldsymbol{\alpha}, \boldsymbol{p})$ and every balance corresponds to the first term $\boldsymbol{\alpha}\tau^{\boldsymbol{p}}$ in an expansion around movable singularities. For such an expansion to describe a *general solution*, the ψ-series must contain $n-1$ arbitrary constants in addition to the arbitrary parameter t_*. The position in the power series where these arbitrary constants appear is given by the *resonances*. Each balance defines a new set of resonances [19]. They are given by the eigenvalues of the matrix $\mathcal{R}$,

$$\mathcal{R} = -D\hat{\boldsymbol{f}}(\boldsymbol{\alpha}) - \mathrm{Diag}(\boldsymbol{p}), \qquad (9)$$

where $D\hat{\boldsymbol{f}}(\boldsymbol{\alpha})$ is the Jacobian matrix evaluated on $\boldsymbol{\alpha}$. The resonances are labeled $r_i, i = 1, \ldots, n$ with $r_1 = -1$. In view of our assumption, the only resonances allowed here are rational. A *general solution* is a formal solution $\boldsymbol{x} = \boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{p}, t)$ with balance $(\boldsymbol{\alpha}, \boldsymbol{p})$ such that $r_j \in \mathbb{Q}^+$ for all $j > 2$.

We can now state the main theorem of this paper:

Theorem. Consider the polynomial system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, a nonlinear polynomial and assume that the general solution can be locally expanded in a convergent ψ-series. Then the two following statements are equivalent:

(a) There exists an open set of initial conditions $X_0 \subseteq \mathbb{R}^n$ such that for all $\boldsymbol{x}_0 \in X_0$, there exists $t_* \in \mathbb{R}$ for which $\lim_{t\to t_*} \| \boldsymbol{x}(t, \boldsymbol{x}_0) \| \to \infty$.

(b) There exists a general solution $\boldsymbol{x} = \boldsymbol{\Psi}(\boldsymbol{\alpha}, \boldsymbol{p}, t; \boldsymbol{c})$ with $\boldsymbol{\alpha} \in \mathbb{R}^n$.

A complete demonstration of this theorem is beyond the scope of this Letter and is given together with other results in Ref. [7]. The main idea is to show that there exists a continuous one-to-one map between the open set of initial conditions and the set of arbitrary constants appearing in the ψ-series. As a consequence, a real open set of initial conditions lead to a real ψ-series that describe the behavior of the solutions around a real singularity.

2.1. The procedure

The procedure to check the occurrence of finite-time blow-up by using the theorem presented here is particularly simple.

The first step consists in finding all the possible balances $(\boldsymbol{\alpha}, \boldsymbol{p})$ by considering all the truncations of the vector field $\boldsymbol{f}$ that satisfy the condition (8). This gives a list of balances $\mathcal{B} = \{(\boldsymbol{\alpha}_i, \boldsymbol{p}_i); i = 1, \dots, m\}$ that correspond to the first term $\boldsymbol{\alpha}\tau^{\boldsymbol{p}}$ in a series around a movable singularity. The problem is that the number m of possible balances can become very large as the dimension n of the system increases. Not all these balances correspond to the expansion of the general solution around a movable singularity.

The second step of the procedure consists in checking which balances defines a general solution. To do so we consider each element of $\mathcal{B}$ and compute the eigenvalues of $\mathcal{R}$ as defined by (9). If the number of positive eigenvalues is $n - 1$ then the balance under consideration corresponds to a general solution. Let $\mathcal{G} \subset \mathcal{B}$ be the set of all balances with such a property: $\mathcal{G} = \{(\boldsymbol{\alpha}, \boldsymbol{p}) \in \mathcal{B} | \mathrm{Spec}(\mathcal{R}) \in (\mathbb{R}^+)^{n-1} \cup \{-1\}\}$.

Finally, the system under consideration will exhibit finite-time blow-up if one of the element of $\mathcal{G}$ is such that $\boldsymbol{\alpha} \in \mathbb{R}^n$. Moreover, as proved in [7], the blow-up will occur for an open set of initial conditions located in the phase-space orthant $\mathrm{sign}(\boldsymbol{\alpha})$.

In the same way, the absence of finite-time blow-up (on open set of initial conditions) will be guaranteed if $\boldsymbol{\alpha} \notin \mathbb{R}^n \ \forall (\boldsymbol{\alpha}, \boldsymbol{p}) \in \mathcal{G}$.

3. An example: the Rikitake system

The model under consideration was first derived by Rikitake in 1958 and consists of two identical single Faraday-disk dynamos of the Bullard type coupled together and is used by geophysicists as a conceptual model to study the time series of geomagnetic polarity reversals over geological time [15]. It reads

$$\dot{x} = -\mu x + zy, \tag{10a}$$

$$\dot{y} = -\mu y + (z - a)x, \tag{10b}$$

$$\dot{z} = 1 - bxy. \tag{10c}$$

The traditional Rikitake model is recovered by setting $b = 1$. We now show that depending on the value of b the solutions of the Rikitake model exhibit (for an open set of initial conditions) finite-time blow-up. To do so, we isolate two possible balances. The first one corresponds to the truncation of the vector field $\widehat{\boldsymbol{f}}_1 = (zy, zx, -bxy)$, with balance

$$\boldsymbol{p}_1 = (-1, -1, -1), \quad \boldsymbol{\alpha}_1 = \left(\frac{\pm i}{\sqrt{b}}, \frac{\pm i}{\sqrt{b}}, 1\right) \text{ or } \boldsymbol{\alpha}_1 = \left(\frac{\pm i}{\sqrt{b}}, \frac{\mp i}{\sqrt{b}}, -1\right). \tag{11}$$

with resonances $-1, 2, 2$. The second balance corresponds to the truncation $\widehat{\boldsymbol{f}}_2 = (zy, -ax, -bxy)$, with balance

$$\boldsymbol{p}_2 = (-2, -1, -2), \quad \boldsymbol{\alpha}_2 = \left(\frac{\pm 2i}{a\sqrt{b}}, \frac{\mp 2i}{\sqrt{b}}, \frac{-2}{a}\right), \tag{12}$$

with resonances $-1, 2, 4$. Therefore, both balances correspond to a general solution.

As a direct application of our theorem, we conclude that if $b > 0$, there is no finite-time blow-up in the Rikitake model (for open set of initial conditions) since no balance $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ are real. However, for $b < 0$

the balances are all real and there exist open sets of initial conditions leading to finite-time blow-up.

Finally, we stress that even when finite-time blow-up is ruled out (for $b > 0$) the solutions are not bounded and can still blow-up in infinite time. Indeed, there is a simple particular solution: $x = y = 0$, $z = t$ with such a property.

4. An application

We now apply our results to a physical system where blow-up is only known to occur numerically but for which the existence of blow-up has never been rigorously demonstrated.

The system in question, introduced by Klapper, Rado and Tabor [16], models ideal three-dimensional incompressible inviscid magnetohydrodynamics. The trace equations for such a system near magnetic null points are defined as follows,

$$T_n = \mathrm{Trace}[(\nabla \boldsymbol{u})^n], \quad P_{n,m} = \mathrm{Trace}[(\nabla \boldsymbol{u})^n (\nabla \boldsymbol{b})^m], \tag{13}$$

where $\boldsymbol{u}$ is the fluid velocity field and $\boldsymbol{b}$ is the magnetic field. The significance of a finite-time singularity in the magnetic field is the associated blow-up in the current which in turn can cause the release of vast amounts of energy: this is, for example, believed to be the mechanism behind the formation of solar flares. The Klapper–Rado–Tabor model gives the evolution of the gradient of the magnetic field whereas the current is given by the curl of the magnetic field. The blow-up of the gradient causes then the curl to blow-up.

We now prove the existence of finite-time singularities using our main theorem.

Written in the variables $(x_1, x_2, x_3, x_4, x_5, x_6) \equiv (T_2, T_3, P_{1,2}, P_{2,2}, P_{1,1}, P_{2,1})$ the model reads

$$\dot{x}_1 = -2x_2 + 2x_3, \tag{14}$$

$$\dot{x}_2 = -\tfrac{1}{2}x_1^2 - \beta x_1 + 3x_4, \tag{15}$$

$$\dot{x}_3 = \tfrac{1}{6}\beta^2 + \tfrac{1}{3}\beta x_1 - x_4, \tag{16}$$

$$\dot{x}_4 = -\tfrac{1}{3}x_1 x_3 + \tfrac{1}{3}\beta x_3 - \tfrac{2}{3}\beta x_2 + \tfrac{2}{3}\gamma x_5, \tag{17}$$

$$\dot{x}_4 = \gamma - x_6, \tag{18}$$

$$\dot{x}_6 = -\tfrac{1}{3}x_1 x_5 + \tfrac{1}{3}\beta x_5. \tag{19}$$

There are several possible balances. However, only one gives a full set of non-negative rational resonances. We find by checking the eigenvalues of $\mathcal{R}$ (see Eq. (9)) that the resonances are $\{-1, 0, 0, 3, 3, 6\}$ for the following balance,

$$\boldsymbol{p} = (-2, -3, -1, -2, -1, -2), \tag{20}$$

$$\boldsymbol{\alpha} = (6, -6, \alpha_3, 2\beta - \alpha_3, \alpha_5, -\alpha_5). \tag{21}$$

The arbitrariness of the leading terms α_3, α_5 reflects the fact that $r = 0$ is a resonance with multiplicity 2. This is a balance that yields a general local solution of the form (7). Note that the leading order coefficients are real, or can be chosen real for the proper initial data. Thus, our theorem predicts that there is an open set of real initial conditions (which make α_3 and α_5 real) for which the general solution to this system blows up in finite time.

Interestingly enough, our result can also be used to give an estimate of the blow-up time. Indeed, close to the singularity the dynamic is controlled by the most dominant part of the vector field (the truncation $\widehat{f}$ of the vector field associated with the given balance). For the system $\dot{\boldsymbol{x}} = \widehat{f}(\boldsymbol{x})$, the two first equations decouple leading to a simple closed system $\ddot{x}_1 = x_1^2$. Therefore, if the system is close enough to the singularity, the blow-up time can be predicted by integrating this system. That is if, for a given time t_1, we know the values of $x_1(t_1)$ and $x_1(t_1)$, then

$$t_* = t_1 + \frac{1}{\sqrt{2E_1}} \lim_{x_1(t)\to\infty} \int_{x_1(t_1)}^{x_1(t)} \frac{\mathrm{d}x_1}{\sqrt{1 + x_1^3/3E_1}}, \tag{22}$$

where $E_1 = \frac{1}{2}\dot{x}_1(t_1)^2 - \frac{1}{3}x_1(t_1)^3$. This last integral can be expressed in terms of an elliptic integral of the first kind,

$$t_* = t_1 + \frac{\sqrt[12]{3}}{\sqrt[6]{8E_1}} F\left(\alpha, \sin\frac{5\pi}{12}\right), \tag{23}$$

where

$$\alpha = \cos^{-1}\left(\frac{1 - \sqrt{3} - x_1(t_1)}{1 + \sqrt{3} - x_1(t_1)}\right).$$

In order to get close to the singularity, we can start with an initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ and compute the value

$x(t_1)$ by expanding the solutions in Taylor series up to a given order N,

$$x(t) = \sum_{i=0}^{N} a_i t^i + O(t^{N+1}), \tag{24}$$

where $a_0 = x_0$ and the values of the coefficients a_i as functions of x_0 can be found by inserting the Taylor series in the system $\dot{x} = f(x)$ and equating power by power. This Taylor series has a finite radius of convergence t_{max} which can be found by studying the Taylor approximations as N increases. We can now choose $t_1 < t_{max}$ and find the value of $x(t_1)$. These values are then used to compute the blow-up time t_* through an estimate like (23). This method is not a general method since it is not guaranteed that t_{max} will be in the radius of convergence of the Ψ-series (indeed t_{max} could be constrained by complex rather than real singularities). However, further approximations of the solutions can be obtained by analytic continuation and/or Padé approximants.

For the system studied here this method provides an excellent approximation of the blow-up time. As an example, we choose the following initial conditions leading to blow-up: $x_0 = (-20, -10, -20, 178/3, 5, 24)$ and $\gamma = 20$, $\beta = 4$. Using successive Taylor approximations with $N \leqslant 25$ we find that $t_{max} \approx 0.55$. Picking $t_1 = 0.5$, we obtain $x_1(t_1) \approx -4.78$, $\dot{x}_1(t_1) \approx 59.23$ and $t_* \approx 1.50$, to be compared with the numerical value $t_{*,num} = 1.54$ obtained by using the dedicated ATOMFT package [39].

5. Conclusions

In this Letter we have show how to detect the generic occurrence or absence of finite-time blow-up in dynamical systems. Here, the absence of finite-time blow-up is understood as *the set of initial conditions leading to a finite-time singularity has zero Lebesgue measure.* This method amounts to checking the real-valuedness of the coefficients of the asymptotic series describing the general solutions. The practical problem is to be able to analyze which asymptotic solutions correspond to the general solution. This is done by checking the number of arbitrary constants in the Ψ-series. The procedure presented here is completely algorithmic and can be performed numerically on large systems of ODEs such as the ones obtained by discretizing PDEs.

An interesting conclusion of this work is that finite-time blow-up is controlled by the most nonlinear terms of the vector field. The lower terms (such as the diagonal linear terms) can create regions in phase space where the solutions are bounded but cannot prevent the solution from blowing up elsewhere in phase- pace.

We have focused our analysis on generic blow-up; the same analysis can be performed to check the blow-up of particular solutions. In turn this is done by carefully checking the asymptotic behavior of these solutions and identifying the corresponding local series around the movable singularities. This analysis will give a complete picture of the possible blow-up in dynamical systems.

Finally, we can use the asymptotic series to find *basins of attractions of singularities.* Indeed the asymptotic *ψ-series* can be used to find initial conditions for which the solution blow-up. These initial conditions can then be continued backwards to obtain global basins in phase-space. This will, hopefully, be the subject of latter work.

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