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THE ROLE OF COMPLEX-TIME SINGULARITIES IN CHAOTIC DYNAMICS

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The analysis of complex-time singularities has proved to be the most useful tool for the analysis of integrable systems. Here, we demonstrate its use in the analysis of chaotic dynamics. First, we show that the Melnikov vector, which gives an estimate of the splitting distance between invariant manifolds, can be given explicitly in terms of local solutions around the complex-time singularities. Second, in the case of exponentially small splitting of invariant manifolds, we obtain sufficient conditions on the vector field for the Melnikov theory to be applicable. These conditions can be obtained algorithmically from the singularity analysis.

1. Introduction

One of the central problem of nonlinear dynamics is the analysis of near-integrable systems. This problem, pioneered in the nineteenth century by such eminent mathematicians as Poincaré, Lyapunov and others, consists in considering the perturbation of a given completely integrable system by some small terms destroying its integrable structure [1]. The problem is then to establish the global dynamics of the perturbed systems and, if possible, predict the onset of chaotic dynamics. In the case of Liouville integrable Hamiltonian systems under perturbation, an understanding of the resulting dynamics is provided by the celebrated KAM theorem [2] which proves, under general conditions on the vector field, the onset of global chaotic dynamics and the persistence of some invariant tori.

While the KAM theorem is mainly concerned with the persistence and destruction of tori consisting of families of *periodic* orbits, the Melnikov theory applies to the perturbation of *homoclinic* or *heteroclinic* orbits — and has been very successful in determining the onset of chaotic dynamics in perturbed integrable systems. Homoclinic manifolds connect invariant sets to themselves and can be thought of as providing the "skeleton" of the phase space since they usually divide it into invariant domains. The onset of chaotic behavior around perturbed homoclinic manifolds was first noticed by Poincaré in the restricted three-body problem, and since then has been one of the central issue in nonlinear dynamics. The importance of homoclinic orbits and of transverse homoclinic intersection (i. e. transverse intersection of the stable and unstable manifolds to the fixed point) can be found in the result of Smale, which generalizes a theorem due to Birkhoff [3]. The Smale-Birkhoff theorem states that the existence of transverse homoclinic intersections for diffeomorphisms is sufficient for the existence of an invariant Cantor set in which the periodic orbits are dense. The work of Birkhoff and Smale on diffeomorphisms highlights the dynamics and geometry of invariant Cantor sets. The

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Matematics Subject Classification 32S70, 34A20 contribution of Melnikov was to give an explicit criterion, based on the so-called *Melnikov function*, for the existence of transverse intersections of invariant manifolds in time-continuous vector fields subject to periodic forcing [4]. We recommend the excellent paper by Chow and Yamashita for an up-to-date study of the Melnikov theory in n-dimensions [5].

What is the possible role of complex-time singularity analysis in chaotic dynamics? It has been known since the early work of Fuchs, Poincaré, Kowalevskaya and Painlevé that the analysis of complex-time series solutions can be used to establish integrability properties of differential equations [6]. For instance, the classification of integrable second order differential equations was established by finding all such equations whose general solutions are meromorphic, the so-called *Painlevé property*. Using this idea, Kovalevskaya was able to find a new integrable case for the Euler equations for rigid body motion, and her identification of the few special cases of integrability spurred Lyapunov to come up with a rigorous proof of its general nonintegrability [7]. (A full account of Kovalevskaya's life can be found P. Kochina's "Love and Mathematics" [8]).

Nowadays, the analysis of the local solutions around their singularities has become a common tool to test for the integrability of ordinary differential equations, partial differential equations and discrete mappings. It has been repeatedly demonstrated that systems exhibiting meromorphic structure have a simple geometric structure in phase space and do not exhibit complex dynamics. However, the converse is not necessarily true since systems with regular dynamics in *real* phase space might not necessarily be meromorphic in *complex* time. It has been recently suggested that the loss of meromorphicity in complex time and the general property of *nonintegrability* might be associated with irregular dynamics in the complexified phase space [9, 10].

We first consider the following model problem: a singularity analysis of integrable systems with given homo/hetero-clinic structure under periodic perturbation, i. e., the traditional setting of Melnikov theory with the additional assumption that the integrable system under investigation has a simple (i. e. meromorphic) singularity structure. This additional assumption implies the existence of local Laurent series around the singularities in complex time. Under perturbation, the meromorphicity is lost and the local solutions may exhibit logarithmic branch points. However, a local expansion analytic in the perturbation parameter can still be developed — this is the the so-called $\Psi-\varepsilon$ expansions [11]. The analysis of these new series, together with the location of singularities for the homoclinic solutions, enables one to compute the Melnikov vector. This analysis thus provides both a transparent relationship between homoclinicity and meromorphicity, and a direct way of computing the Melnikov vector and predicting the onset of chaotic motion in perturbed systems.

The second problem we consider is closely related to the first. Here again, we consider near-integrable systems. The main difference is in the perturbation: we now allow for periodic forcing whose period vanishes with the perturbation parameter. The geometric effect of such a perturbation is to split the invariant manifolds by a distance that is exponentially small in the perturbation parameter. The problem of computing such a distance was first outlined by Poincaré and has been shown to appear in a number of generic problems in nonlinear dynamics such as the averaging of perturbation [12], the discretization of continuous vector fields [13], the divergence of Poincaré-Birkhoff normal forms [2] and the splitting of heteroclinic cycles in KAM theory [14]. The general assumptions of the Melnikov theory are not satisfied in the case of exponentially small splitting (namely, the splitting distance is not analytic in the perturbation parameter), and a general theory to compute such effects has proved elusive. Nonetheless, in the last decade, a number of special cases have been studied and additional conditions on the perturbation (namely, on the ratio of amplitude to period) can be derived to ensure that the Melnikov vector still provides a valid estimate of the splitting distance [15]. We show that these conditions are, in fact, intimately related to the singularity structure of the vector field and of the perturbation, and can be obtained for general n-dimensional systems. The rather lengthy technical details of this work are given elsewhere [16, 17].

2. General setup

We now specify the class of systems to which our results can be applied. Rather than studying the most general case, we introduce, for the sake of clarity, a class of nonlinear vector fields most frequently encountered in nonlinear problems. Namely, a system of ordinary differential equations in $\mathbb{R}^n$ under a periodic perturbation $\mathbf{g}(\mathbf{x},t)$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x},t), \qquad \mathbf{x} \in \mathbb{R}^n \tag{1}$$

with the assumptions:

A1 The vector field $\mathbf{f}(\mathbf{x})$ is polynomial and volume preserving ($\mathrm{Tr}(D\mathbf{f}) = 0$).

A2 There exists a homoclinic manifold to the fixed point $x = 0$ of dimension m. Every homoclinic solution $\widehat{x}(t, t_0)$ is periodic along the imaginary axis with period $\omega > 0$ (i. e. $\widehat{\mathbf{x}}(t + i\omega, t_0) = \widehat{\mathbf{x}}(t, t_0)$) and has a finite number of isolated singularities in the complex strip $\mathrm{Im}(t) < \omega$ where it is meromorphic. That is, around the movable singularity t_*, there exists a series solution of the form:

$$\mathbf{x} = \alpha (t - t_*)^{\mathbf{p}} \sum_{j=1}^{\infty} \mathbf{a}_j (t - t_*)^j \tag{2}$$

with $\alpha, \mathbf{a}_j \in \mathbb{C}^n\ \forall\, j$; $\mathbf{a}_0 = 1$; $\mathbf{p} \in \mathbb{Z}^n$.

A3 The perturbation $\mathbf{g}(\mathbf{x},t)$ is conservative in $\mathbf{x}$ and periodic in t of period $T\varepsilon^{\gamma}$ and of zero mean. If $\gamma > 0$, then the perturbation is referred to as a *singular perturbation*, *regular* otherwise. Moreover, the perturbation is assumed to be non-dominant around the singularity t_*. That is, there exists a vector $\mathbf{q} \in \mathbb{N}_0^n$, such that:

$$\mathbf{g}(\mathbf{x}(t), t) \underset{t \to t_*}{\sim} \mathbf{c}(t - t_*)^{\mathbf{p}+\mathbf{q}-1}, \tag{3}$$

with $\mathbf{c} \in \mathbb{C}_0^n$.

2.1. The Melnikov vector

The Melnikov vector gives a measure of the splitting distance between stable and unstable manifolds in different directions of the phase space. In order to define the Melnikov vector we introduce the *variational equation* together with the *adjoint variational equation*:

$$\dot{\mathbf{u}} = D\mathbf{f}(\widehat{\mathbf{x}})\mathbf{u}, \qquad \dot{\bar{\mathbf{u}}} = -\bar{\mathbf{u}} D\mathbf{f}(\widehat{\mathbf{x}}), \tag{4}$$

where $D\mathbf{f}(\widehat{\mathbf{x}})$ is the Jacobian matrix evaluated on the m-dimensional homoclinic solution.

Proposition 1. *There exists a fundamental solution* $\mathbf{Q}$ *(resp.* $\bar{\mathbf{Q}}$*) of the (resp. adjoint) variational equation, whose columns (resp. rows) forms a set of linearly independent solutions satisfying:*

(i) *The rows* $\bar{\mathbf{Q}}_1, \ldots, \bar{\mathbf{Q}}_m$ *are bounded in* t *for some* $1 \le m \le n-1$.

(ii) $\mathbf{Q}\,\bar{\mathbf{Q}} = \mathbf{1}$.

(iii) *Locally, around the singularity* t_* *of* $\widehat{\mathbf{x}}$, *the column (resp. rows) vectors of the fundamental solutions* $\mathbf{Q}$ *(resp.* $\bar{\mathbf{Q}}$*) behaves as follows:*

$$\mathbf{Q}_i(t) \underset{t \to t_*}{\sim} \boldsymbol{\beta}_i (t - t_*)^{\mathbf{p}+r_i}, \quad \bar{\mathbf{Q}}_i(t) \underset{t \to t_*}{\sim} \bar{\boldsymbol{\beta}}_i (t - t_*)^{-\mathbf{p}-r_i}, \quad i = 1, \ldots, n \tag{5}$$

where Let $\mathbf{R} = D\mathbf{f}(\alpha) - \mathrm{diag}(\mathbf{p})$, *then* $\boldsymbol{\beta}_i$ *(resp.* $\bar{\boldsymbol{\beta}}_i$*) is an eigenvector of* $\mathbf{R}$ *(resp.* $\mathbf{R}^T$*) of eigenvalue* r_i.

The exponents r_i appearing in (5) are called the *resonances* and in many instances can be identified with the *Kowalevskaya exponents* [18]. They will play a fundamental role hereafter. Let $\widetilde{\mathbf{Q}}$ be the $m \times n$ matrix whose rows are $\bar{\mathbf{Q}}_1, \ldots, \bar{\mathbf{Q}}_m$. The Melnikov vector is then defined with respect to the bounded solutions of the adjoint variational equation:

Definition 1. The Melnikov vector $\mathcal{M}(t_0) : \mathbb{R} \to \mathbb{R}^m$:

$$\mathcal{M}(t_0) = \int_{-\infty}^{+\infty} \widetilde{\mathbf{Q}}\mathbf{g}(\hat{\mathbf{x}}(t), t + t_0)\, dt. \tag{6}$$

The Melnikov vector can be used to detect the occurrence of homoclinic intersections in the perturbed dynamics of system (1) in the case of regular perturbation [5]:

Proposition 2. *The stable and unstable manifolds of (1) to* $\mathbf{x} = 0$ *exhibit, for regular perturbation and for* ε *small enough, transverse homoclinic intersections if there exists* $t_1 \in \mathbb{R}$ *and* $1 \leq i \leq m$ *such that:*

$$\mathcal{M}(t_1) = 0 \qquad \text{and} \qquad \frac{d\mathcal{M}}{dt_0}(t_1) \neq 0. \tag{7}$$

However, if the perturbation is singular (that is, in the case of fast forcing), then no such results can be established in general and additional assumptions must be imposed on the perturbation for this result to hold. The case of singular perturbation and exponentially small splitting will be discussed in Section 4.

3. Singularity analysis and the Melnikov vector: the regular case

We now show how to obtain the Melnikov vector by using the singularity analysis for regular perturbation. In this case, one can always introduce a change of variable such that the amplitude of the perturbation is independent of the perturbation parameter, that is, $\gamma = 0$. To give an explicit estimate of the splitting distance we combine two main results. First, we reduce the Melnikov vector to a contour integral that can be computed by the method of residues. Second, we show the existence of local solutions (the so-called $\Psi - \varepsilon$ series) around the singularities of the homoclinic solution for the perturbed problem. Finally, we show that the residues of the Melnikov vector are related to the coefficients of the $\Psi - \varepsilon$ series.

3.1. The residues of the Melnikov vector

It is possible to evaluate the Melnikov vector by the method of residues. Let us Fourier expand $\mathbf{g}(\widehat{\mathbf{x}}(t), t + t_0)$ as well as the Melnikov vector $\mathcal{M}(t_0)$:

$$\mathbf{g}(\widehat{\mathbf{x}}(t), t + t_0) = \sum_{k=-\infty}^{+\infty} \mathbf{g}^{(k)}(\widehat{\mathbf{x}}) e^{ik(t+t_0)}, \tag{8}$$

$$\mathcal{M}(t_0) = \sum_{k=-\infty}^{+\infty} \mathcal{M}^{(k)} e^{ikt_0}, \tag{9}$$

$$\mathcal{M}^{(k)} = \int_{-\infty}^{+\infty} \widetilde{\mathbf{Q}}\mathbf{g}^{(k)}(\widehat{\mathbf{x}}(t)) e^{ikt}. \tag{10}$$

For $k \neq 0$ and using assumption A2, $\mathcal{M}^{(k)}$ can be evaluated:

Proposition 3. *The Melnikov vector is:*

$$\mathcal{M}(t_0) = \mathcal{M}^{(0)} + \sum_{k \neq 0} \frac{2\pi i}{(1 - e^{-\omega k})} \sum_{t_* \,\in\, S} \operatorname*{res}_{t_*}(\widetilde{\mathbf{Q}}\mathbf{g}^{(k)}(\widehat{x}) e^{ikt}), \tag{11}$$

where $S = \{t \in \mathbb{C};\ 0 \le \mathrm{Im}(t) \le \omega\}$ *is the fundamental domain of the homoclinic solution.*

This result is obtained by taking a closed contour following the domain S in the complex plane and computing the shift between the two contributions of the integrals on the real axis and on a path parallel to the real axis passing through the point $t = 0 + i\omega$ in the complex plane. A similar result has been obtained for the planar Hamiltonian case by Ziglin [19].

3.2. Singularity analysis around the perturbed homoclinic solution

The homoclinic solution is, as a consequence of assumption A2, meromorphic around its singularities. Therefore, it can be expanded into a Laurent series solution (2). This series is characterized by its *dominant behavior* $\mathbf{p} \in \mathbb{Z}^n$ and its *resonances* (the resonances are the Fuch's indices of the variational equations around the Laurent series, that is the elements of Spec$\{\mathbf{R}\}$).

Let us now consider the perturbed system (1). This system is, in general, nonintegrable and the meromorphicity of the homoclinic solutions is destroyed by the perturbation. However, there exists a more general series solutions:

Proposition 4. *Under assumptions A1, A2, A3, the nonintegrable system (1) admits, for regular perturbation, a formal* $\Psi - \varepsilon$ *expansion for the perturbed homoclinic solution:*

$$\widehat{\mathbf{x}} = (t - t_*)^{\mathbf{p}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \mathbf{a}_{ijk} (t - t_*)^i \varepsilon^j Z^k, \tag{12}$$

where $Z = \log(t - t_*)$.

A more convenient form for these ε-expansion can be given in terms of a power series in (ε, Z) whose coefficient are Laurent series:

Proposition 5. *Under assumptions A1 and A3, the nonintegrable system (1) admits a formal* ε*-expansion for the perturbed homoclinic solution:*

$$\widehat{\mathbf{x}} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \mathbf{s}_{ij} \varepsilon^i Z^j, \tag{13}$$

where $Z = \log(t - t_*)$ *and* $\mathbf{s}_{ij}$ *are convergent Laurent series in a punctured disk centered in* t_*.

These series have many interesting features. At each order in ε, there is only a finite number of logarithmic terms, and all the coefficients are convergent Laurent series. This is a great simplification of the more cumbersome procedure needed to compute the usual Ψ-series [20]. The existence of the $\Psi - \varepsilon$ series as an actual solution remains open. For our analysis, this is not a problem since the information needed to compute the Melnikov vector is contained at $\mathcal{O}(\varepsilon)$ and the residues are still well-defined.

3.3. The Melnikov vector

Under the assumptions A1—A3, the Melnikov integral can be computed in terms of residues and the $\Psi - \varepsilon$ series can be derived. We now combine these results and show that the Melnikov integral can be estimated by the coefficients of the $\Psi - \varepsilon$ expansions.

Proposition 6. *Under assumptions A1 to A3, the Melnikov vector is given by*

$$\mathcal{M}(t_0) = \mathcal{M}^{(0)} + 2\pi i \sum_{k} \frac{e^{ikt_0}}{1 - e^{-\omega k}} \sum_{t_* \in S} \mathcal{K}^{(k)}(t_*) e^{ikt_*}, \tag{14}$$

where $\mathcal{K}^{(k)}(t_*)$ *is the k-th component of the Fourier expansion of* $\mathcal{K}(t_0,t_*)$ *w.r.t.* t_0 *computed from the coefficients of (12) and the eigenvectors of* $\mathbf{R}^T$*:*

$$\mathcal{K}_i(t_0,t_*) = \bar{\boldsymbol{\beta}}_i \mathbf{a}_{r_i,1,1} \qquad i = 1,\dots,m, \tag{15}$$

where $\mathbf{a}_{r_i,1,1}$ *is a coefficient of the series (12).*

This last proposition gives an explicit and algorithmic way of computing the Melnikov vector in terms of the local series solutions. Explicit examples are given in the last section of this paper. For $k = 0$, the result does not hold, but for that particular case, the classic result from Andronov [21] can be applied in two dimensions and the Melnikov integral can be found again by the method of residues. In higher dimensions the zero component of the Melnikov vector can still be obtained by a local analysis of the adjoint variational equations [11].

4. Singularity analysis and the Melnikov vector: the singular case

We now turn to the case of singular perturbations. This arises, when the system is under fast forcing, that is when the period vanishes with the perturbation parameter. Explicitly, we consider systems of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon \mathbf{g}(\mathbf{x}, \frac{t}{\mu}), \tag{16}$$

where, eventually, ε will be taken as a power of μ. In this case, the splitting distance becomes exponentially small with the perturbation parameter. To illustrate this phenomenon, we consider on Fig. 1 and Fig. 2, the Duffing equation in the regular (slow forcing) and singular case (fast forcing). As explained in the two previous sections, if $\mu = 1$, the Melnikov vector gives an approximation of the splitting distance between invariant manifolds, that is

$$\mathcal{D}(t_0) = \varepsilon^\gamma \mathcal{M}(t_0) + \mathcal{O}(\varepsilon^2). \tag{17}$$

If, however $\mu \ll 1$, then the splitting distance becomes exponentially small. To illustrate the problem, consider the forced pendulum, a paradigmatic example [15]:

$$\ddot{x} + \sin(x) = \varepsilon \sin(\frac{t}{\mu}). \tag{18}$$

The evaluation of the Melnikov integral (that is the Melnikov vector with $m = 1$) is straightforward:

$$\mathcal{M}(t_0) = 2\pi \mathrm{sech}(\frac{\pi}{2\mu}) \sin(\frac{t_0}{\mu}) \tag{19}$$

so that the splitting distance becomes:

$$d(t_0) = \varepsilon \pi \mathrm{sech}(\frac{\pi}{2\mu}) + \mathcal{O}(\varepsilon^2). \tag{20}$$

In order for this estimate to be valid one must have $\varepsilon \sim \mathcal{O}(e^{-\frac{\pi}{2\mu}})$ so that if $\varepsilon \to 0$ as $\mu \to 0$, μ must also be exponentially small.

One way to circumvent this problem is to obtain better estimate on the error term $\mathcal{O}(\varepsilon^2)$ and show that this term is itself exponentially small in μ. The main difficulty in doing so is that this property is not true in general [22] and conditions on ε as a function of μ have to be introduced. More specifically, one sets $\varepsilon = \mu^\gamma$ and looks for conditions on γ for the Melnikov vector to provide a valid estimate of the splitting distance. In the case of the forced pendulum, successive improved bounds have been proposed in the last decade by different authors: $\gamma \leq 8$ [22]; $\gamma > 5$ [23]; $\gamma \leq 3$ [24]; $\gamma \leq 0$ [15].

The best bound was obtained by Delshams and Seara. Subsequently they developed in the case of planar conservative systems a general method to provide sufficient conditions on γ for the Melnikov

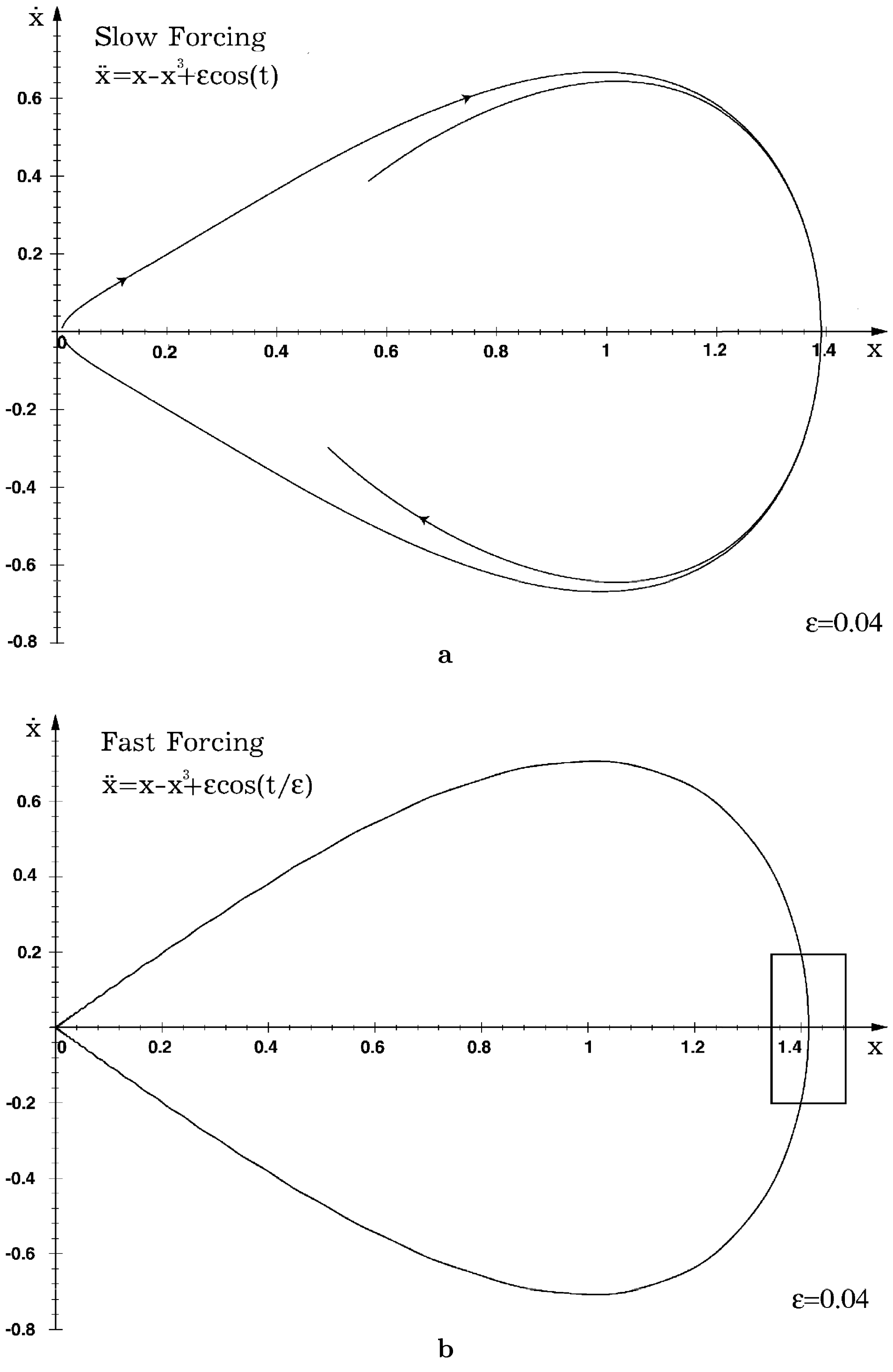

Fig.1 (a) The regular case: the splitting distance is linear in the perturbation parameter. (b) The singular case, the splitting can only be seen by repeated zooming (see Fig 2).

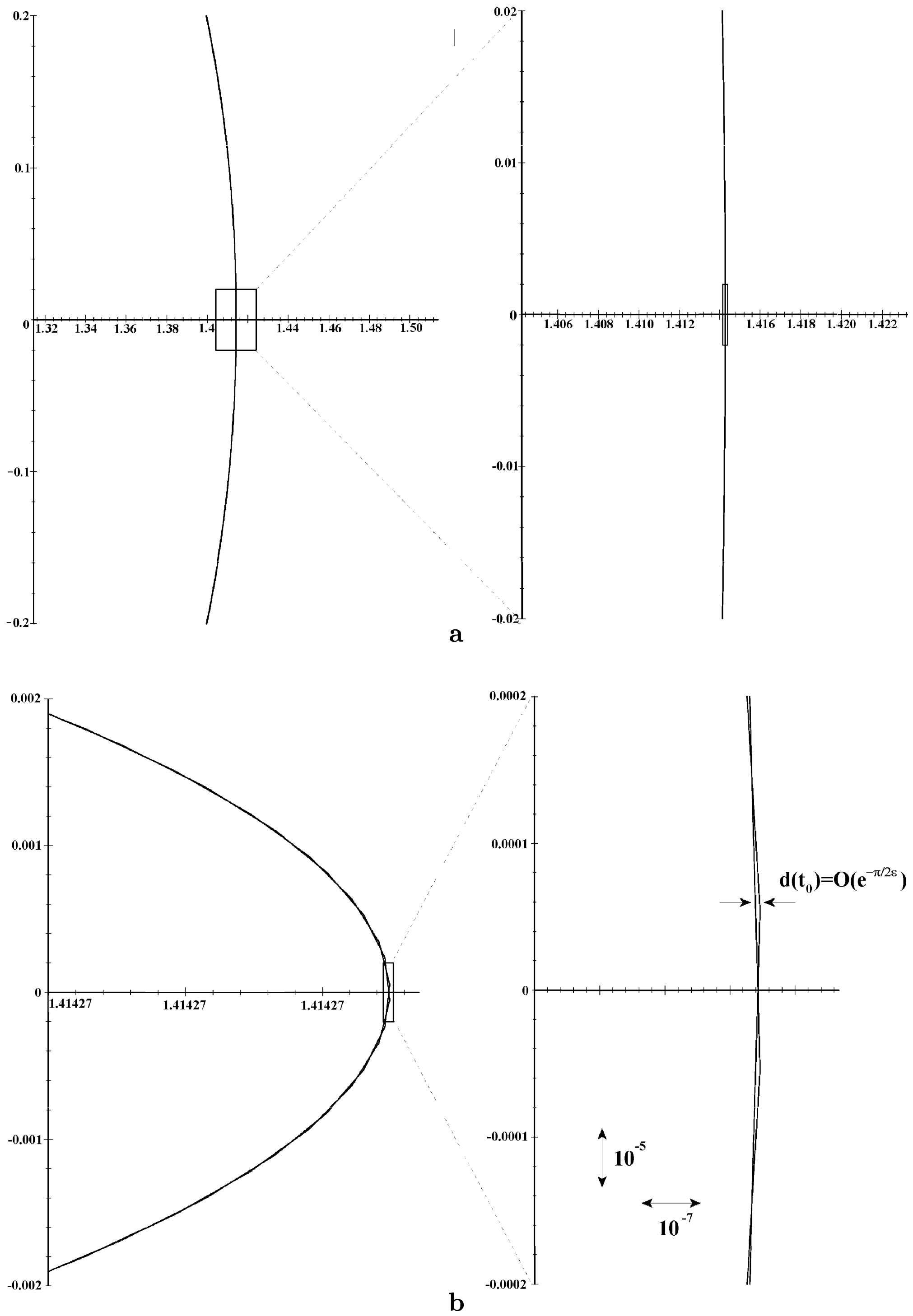

Fig.2 Repeated zooming shows the exponentially small splitting of the separatrix. The computation was performed with 120 digit accuracy.

function to be a valid estimate of the splitting distance. We build on their work together with the formalism and the methods introduced in [11] to propose a general bound for n-dimensional systems.

Proposition 7. *Consider the system (16) with* $\varepsilon = \mu^\gamma$ *and assumptions A1, A2, A3. Let* $q_{\min} = \min\{q_1, \dots, q_n\}$, $r_{\max} = \max\{r_1, \dots, r_n\}$ *defined by the local series (2,3,5). Then, provided that* $\gamma \geq \{0, r_{\max} - q_{\min}\}$, *the Melnikov vector (6) gives a valid leading order approximation of the splitting distance* $\mathcal{D}$. *That is,*

$$\mathcal{D}(t_0) \sim \mathcal{M}(t_0)\left[1 + \mathcal{O}(\mu\varepsilon^\lambda)\right] \qquad \forall\, t_0 \in \mathbb{R} \tag{21}$$

with $\lambda > 0$.

The remarkable feature of this result is that it only depends on the local behavior of the solutions around the singularities and is straightforward to check. The complete proof of this result is rather long and is given in complete details in [16, 17]. Here we present a rough outline of the proof in order to give to the interested reader a general idea of the techniques involved.

Outline of the proof The proof involves the following series of steps.
1. The main idea of the proof is to explore the behavior of the splitting distance in complex time. That is we first show that

$$\mathcal{D}(t_0) \sim \mathcal{M}(t_0)[1 + \mathcal{O}(\varepsilon\mu^\lambda)], \tag{22}$$

$$\mathcal{D}(t_0 \pm t_* - i\mu) \sim \mathcal{M}(t_0 \pm t_* - i\mu))\left[1 + \mathcal{O}(\varepsilon\mu^\lambda)\right] \text{ with } \lambda > 0. \tag{23}$$

2. We define the distance between the stable and unstable manifold $x^{u,s}(t,z)$ of the perturbed system and the unperturbed homoclinic solution $\widehat{\mathbf{x}}$:

$$\boldsymbol{\xi}^{u,s}(t,z) = \mathbf{x}^{u,s}(t,z) - \widehat{\mathbf{x}}(t+z) \tag{24}$$

for $t, z \in \mathbb{C}$. It can be shown that $\boldsymbol{\xi}$ satisfies the following equation:

$$\dot{\boldsymbol{\xi}}(t,z) = D\mathbf{f}(\widehat{\mathbf{x}})\boldsymbol{\xi} + \varepsilon\mu^\lambda \mathbf{g}\left(\widehat{\mathbf{x}}(t+z), \frac{t}{\mu}\right) + \widetilde{F}(\boldsymbol{\xi}, t+z, \frac{t}{\mu}) \tag{25}$$

where

$$\widetilde{F}(\boldsymbol{\xi}, t+z, \frac{t}{\mu}) = f(\widehat{\mathbf{x}} + \boldsymbol{\xi}) - f(\widehat{\mathbf{x}}) - Df(\widehat{\mathbf{x}})\boldsymbol{\xi} + \varepsilon\mu^\lambda \left[\mathbf{g}\left(\widehat{\mathbf{x}} + \boldsymbol{\xi}, \frac{t}{\mu}\right) - \mathbf{g}\left(\widehat{\mathbf{x}}, \frac{t}{\mu}\right)\right].$$

3. The core of the proof is then to build a *recursive integral equations* for $\boldsymbol{\xi}$, this directly follows from the work of Delshams and Seara [25]:

$$\begin{aligned}
\boldsymbol{\xi}_1(t,z) &= \mathbf{Q}(t+z)\int_{\pm\infty}^{t} \mathbf{Q}^{-1}(\sigma+z)\varepsilon\mu^\lambda \mathbf{g}\left(\widehat{\mathbf{x}}(\sigma+z), \frac{\sigma}{\mu}\right) d\sigma,\\
\boldsymbol{\xi}_{k+1}(t,z) &= \boldsymbol{\xi}_1(t,z) + \mathbf{Q}(t+z)\int_{\pm\infty}^{t} \mathbf{Q}^{-1}(\sigma+z)\widetilde{F}\left(\boldsymbol{\xi}_k, \sigma+z, \frac{\sigma}{\mu}\right) d\sigma.
\end{aligned}$$

4. It is now possible to define the splitting vector:

$$\mathcal{D}(t_0) = \lim_{k\to\infty} \widehat{\mathbf{Q}}\left(\boldsymbol{\xi}_k^u(-t_0, z) - \boldsymbol{\xi}_k^s(-t_0, z)\right). \tag{26}$$

Note, that if $k = 1$, the above expression reduces to the Melnikov vector $\mathcal{M}(t_0)$.

5. Show that $\mathbf{Q}, \bar{\mathbf{Q}}, \mathbf{g}$ are bounded in

$$\mathcal{S} = \{(t, z) : |t + \mathcal{I}m(z)| \leq t_* - \mu \text{ and } |t + \mathcal{R}e(z)| \leq T < \infty\}. \tag{27}$$

6. These bounds are given by the local behavior around the singularities:

$$|\mathbf{Q}_j| \leq K\tau^{-\mathbf{p}+r_j}, \qquad |\bar{\mathbf{Q}}_j| \leq K\tau^{\mathbf{p}-r_j}, \qquad |\mathbf{g}(\widehat{\mathbf{x}}, \frac{t}{\mu})| \leq K\tau^{-\mathbf{p}-1+\mathbf{q}}, \tag{28}$$

where $\tau \equiv |t + t_0 \pm t_*|$ and K is a constant.

7. In the domain $\mathcal{S}$ and provided that $\lambda = \gamma - r_{\max} + q_{\min}$, we have:

$$\|\boldsymbol{\xi}^{u,s}_{n+1} - \boldsymbol{\xi}^{u,s}_{n}\|_\infty \leq K\mu^\lambda \|\boldsymbol{\xi}^{u,s}_{n} - \boldsymbol{\xi}^{u,s}_{n-1}\|_\infty. \tag{29}$$

8. So that, the Melnikov vector in $\mathcal{S}$ behaves as follows:

$$|\mathcal{M}_j(t_0 \pm t_* - i\mu)| \sim |\bar{\mathbf{Q}}_j \cdot \boldsymbol{\xi}_1(-t_0, z)| \sim \varepsilon\mu^{\gamma - r_{\max} + q_{\min}}. \tag{30}$$

9. The same type of estimate can be extended to all n:

$$|\bar{\mathbf{Q}}_j \cdot \boldsymbol{\xi}_k(-t_0, z)| \leq |\bar{\mathbf{Q}}_j \cdot \boldsymbol{\xi}_1(-t_0, z)| \left[1 + \mathcal{O}(\varepsilon\mu^\gamma)\right]. \tag{31}$$

10. Therefore, if $\gamma \geq \max\{0, r_{\max} - q_{\min}\}$ we have

$$\boldsymbol{\xi}_k \to \boldsymbol{\xi} \quad \text{and} \quad \mathcal{D}(z) \sim \mathcal{M}(z) \left[1 + \mathcal{O}(\varepsilon\mu^\lambda)\right], \tag{32}$$

with $\lambda > 0$.

5. Examples

5.1. The low and high energy ABC flow

The ABC flow is a well-know dynamical system used in fluid mechanics. It reads [26, 27]:

$$\dot{x} = A\sin z + C\cos y, \quad \dot{y} = B\sin x + A\cos z, \quad \dot{z} = C\sin y + B\cos x.$$

Here, $A > B > 0$, $C = \delta\varepsilon^\lambda$.

For $C = 0$, there is a conserved quantity: $H \equiv A\sin x + B\cos z = h/\varepsilon$. It can be used to simplify the system by setting $y = \dfrac{ht}{\mu}$. After the change of variables: $(q_1, p_1, q_2, p_2) = (B\cos x, B\sin x, A\cos z, A\sin z)$, the system reads:

$$\dot{q}_1 = -p_1(p_2 + \varepsilon\cos(ht/\mu)), \qquad \dot{p}_1 = q_1(p_2 + \varepsilon\cos(ht/\mu)), \tag{33}$$

$$\dot{q}_2 = -p_2(q_1 + \varepsilon\sin(ht/\mu)), \qquad \dot{p}_2 = q_2(q_1 + \varepsilon\sin(ht/\mu)). \tag{34}$$

For low energy (μ of order one), the system can be solved via the regular Melnikov theory. Indeed for $\varepsilon = 0$, this system has three first integrals:

$$p_1 + q_2 = H, \qquad q_1^2 + p_1^2 = B^2, \qquad q_2^2 + p_2^2 = A^2. \tag{35}$$

And a homoclinic orbit:

$$q_1 = \frac{-2ab\sinh(at)}{b^2+\cosh^2(at)}, \qquad p_1 = B\frac{b^2+1-\sinh^2(at)}{b^2+\cosh^2(at)}, \tag{36}$$

$$q_2 = A\frac{-b^2+\cosh^2(at)}{b^2+\cosh^2(at)}, \qquad p_2 = A\frac{2b\cosh(at)}{b^2+\cosh^2(at)}, \tag{37}$$

where $a^2 = AB,\ b^2 = \dfrac{B}{A-B}$.

We can now perform the singularity analysis and obtain the Melnikov vector by using (14). It proceed as follows:

•**The singularity analysis:** There are four singularities in S:

$$t_*^{(1,2)} = \frac{1}{2}\left(i\pi \pm \log\left(\frac{c+1}{c-1}\right)\right), \qquad t_*^{(3,4)} = \frac{1}{2}\left(3i\pi \pm \log\left(\frac{c+1}{c-1}\right)\right),$$

with $c^2 = A/B$.

• **Around the singularity:** we can find the dominant balance: $\mathbf{x} \sim (t-t_*)^{\mathbf{p}}$

$$p = (-1,-1,-1,-1), \quad q = (2,2,2,2), \quad r \in \{-1,1,2,2\}. \tag{38}$$

•**The Melnikov vector:** Knowing the position of the singularities and the coefficient of the $\Psi - \varepsilon$ series, the Melnikov vector is readily found to be:

$$M_1(t_0) = \sqrt{2}\pi(\cos\xi - \sin\xi)\operatorname{sech}(\frac{h\pi}{2\mu a})\sin t_0, \tag{39}$$

$$M_2 = 0, \tag{40}$$

$$M_3 = 0, \tag{41}$$

where $\xi = \dfrac{h}{2a}\log\dfrac{c+1}{c-1}$.

We now look at the high energy case, in particular the case where $\mu = \varepsilon^\gamma$. That is, we look in the space of parameters for regions where the Melnikov estimate for the splitting distance is still valid. As a direct consequence of the previous chapter, we obtain the sufficient condition:

$$\gamma \geq \max\{0, q_{\min} - r_{\max}\} = 0. \tag{42}$$

5.2. Homogeneous integrable Hamiltonians under perturbation

Let us consider a Liouville integrable homogeneous Hamiltonian with n degrees of freedom in the symplectic variables $\mathbf{z} = (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n}$ with diagonal kinetic part under a periodic perturbation:

$$H(\mathbf{x},\mathbf{y}) = H_0(\mathbf{x},\mathbf{y}) + \varepsilon H_1(\mathbf{x},\frac{t}{\mu}), \tag{43}$$

where $H_0(\mathbf{x},\mathbf{y}) = \frac{1}{2}\mathbf{y}.\mathbf{y} + V_0(\mathbf{x})$ with $V_0(\mathbf{x})$ homogeneous of degree d_0.

The unperturbed Hamiltonian H_0 is weight-homogeneous, that is there exists $a, b \in \mathbb{R}$ such that $H_0(\kappa^a\mathbf{x}, \kappa^b\mathbf{y}) = \kappa^{\rho_1}H_0(\mathbf{x},\mathbf{y})$ for all $\kappa \in \mathbb{R}_0$. The weights a, b are readily found to be: $a = \dfrac{d_0}{2-d_0}$ and $b = \dfrac{d_0}{2-d_0}$.

Since the system is Liouville integrable, we can choose the first integrals $J_1 = H_0, \ldots, J_n$ to be also weight-homogeneous with degrees $\rho_1, \ldots, \rho_n$ with respect to the same weights. For instance, $\rho_1 = \dfrac{2d_0}{d_0-1}$. Let $\widehat{H}_1$ be the homogeneous component of highest degree d_1 of H_1.

In addition, we assume that H_0 has a n-dimensional homoclinic manifold $\widehat{x} = \widehat{x}(t)$ built on the zero level set of $J_1 = H_0, \dots, J_n$. This is a generic case for Liouville integrable systems.

In such a situation, the gradients of the first integrals provide a natural basis for the space of bounded solutions of the adjoint variational equation. The Melnikov vector reads then:

$$\mathcal{M}_i(t_0) = \int_{-\infty}^{+\infty} \{J_i, H_1\}(\widehat{x}(t), \frac{t+t_0}{\mu})dt, \qquad i = 1, \dots, n, \tag{44}$$

where $\{\cdot,\cdot\}$ is the Poisson brackets.

Now, we let $\varepsilon = \mu^\gamma$ and look for conditions on γ for the Melnikov vector to give the correct estimate for the splitting distance. To do so, one needs to compute the vector $\mathbf{q}$ in (3) and the resonance set $\{r_1, \dots, r_n\}$. This computation is greatly simplified by the fact that the potential is homogeneous. Indeed the dominant behavior of the homoclinic solution around a movable singularities is $\mathbf{z} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ is $\mathbf{p} = (-a, \dots, -a, -b, \dots, -b)$. The vector $\mathbf{q}$ is then given by

$$\mathbf{q} = \left(\frac{d_0}{d_0-2}, \dots, \frac{d_0}{d_0-2}, \frac{2(d_0-d_1)}{d_0-2}, \dots, \frac{2(d_0-d_1)}{d_0-2}\right) \tag{45}$$

and $q_{\min} = \dfrac{2(d_0-d_1)}{d_0-2}$.

The resonances can be computed directly since the weighted degrees of the first integrals are the Kowalevskaya exponents [28, 29, 18]. Hence, $r_{\max} = \max\{\rho_1, \dots, \rho_n\}$.

Therefore, the Melnikov vector will provide the desired estimate if

$$\gamma \geq \max\{0, \frac{2(d_0-d_1)}{d_0-2} - r_{\max}\}. \tag{46}$$

This provides a simple transparent criterion for Hamiltonian systems simply based on the degrees of the Hamiltonian, the perturbation and the first integrals.

More general systems such as non-homogeneous Hamiltonian systems can be dealt with along the same lines but extra care should be taken as the number of possible Ψ-series increases rapidly in this more general setting.

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РОЛЬ ОСОБЕННОСТЕЙ В КОМПЛЕКСНОМ ВРЕМЕНИ В ХАОТИЧЕСКОЙ ДИНАМИКЕ

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Доказано, что анализ особенностей в комплексном времени является наиболее полезным инструментом для анализа интегрируемых систем. Здесь демонстрируется его применение в хаотической динамике. Во-первых, показано, что вектор Мельникова, дающий оценку расстоянию между инвариантными многообразиями при их расщеплении, может быть явно выражен через локальные решения около особенностей в комплексном времени. Во-вторых, в случае экспоненциально малого расщепления инвариантных многообразий получены достаточные условия на векторное поле для применимости теории Мельникова. Эти условия могут быть получены алгоритмически из анализа сингулярностей.