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The Dynamics of Stretchable Rods in the Inertial Case

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Abstract. Nonlinear amplitude equations for the near-threshold behavior of twisted extensible elastic rods under tension with inertial and dissipative dynamics are derived. In the inertial case localized solutions to the amplitude equations are derived and a linear stability criterion for the pulse solutions is obtained using the Hamiltonian formulation of the problem.

Key words: elastic rods, nonlinear analysis, linear stability, orbital stability, Hamiltonian structure

1. Introduction

The Kirchhoff equations for elastic rods stand as a landmark in the development of classical continuum mechanics [1, 2]. The static versions of these equations have been studied in the greatest detail since the nineteenth century with a multitude of applications to physical and engineering problems; while more contemporary applications have, for example, been concerned with modelling conformational geometries of DNA [3–13]. The study of the full dynamical equations, either numerically or analytically, presents much greater challenges and it is only in more recent times that progress has been made in this context. The inclusion of "extensibility", i.e., that the rod is stretchable, in the Kirchhoff model is classical but, unlike the static case, it further complicates the analysis of the dynamical problem, and in many cases the physically less realistic inextensible case has proved to be a convenient starting point for dynamical studies. Over the past few years the current authors have derived nonlinear amplitude equations to describe the dynamics of inextensible twisted elastic rods beyond the bifurcation threshold [14–17]. These equations have been able to describe a variety of buckling phenomena ranging from the spontaneous change of handedness ("perversion") seen in the tendrils of climbing plants, to the self-assembly of bacterial filaments [18, 19]. Missing, however, from these recent dynamical studies have been two important components: the inclusion of extensibility in the amplitude equations, and an investigation of its effect on the stability of the solutions to these equations. This paper is mainly concerned with a detailed examination of the stability issue for extensible rods in the inertial case, and represents an extension of some recent work by one of us [20, 21]. Our presentation will have an expository flavor. Although the formulation of Kirchhoff's equations for the extensible case has been the subject of careful study many years ago [1] we will give a statement of the key ideas before summarizing the associated amplitude equations. For completeness we show the amplitude equations for both inertial and dissipative (i.e., a locally damped rod) cases, and then proceed to an account of the stability analysis, although this is restricted to the inertial case since the technique is based on the Hamiltonian structure of the equations. Finally we mention that extensibility is but one of a number of important effects that can be included in the Kirchhoff model. Others include shearability [2, 22], hemitropy (the coupling of extension and twist) [23], and – of great importance in the context of biological problems – growth.

2. Kirchhoff Equations for an Extensible Rod

2.1. Arc-Length vs. Material Parameterization

In the description of inextensible rods it is usual to work with arc-length parameterization. Thus a given space curve, $\mathbf{x}$, is parameterized by the arc-length s and, where appropriate, time t, i.e. $\mathbf{x} = \mathbf{x}(s, t)$. The associated unit tangent vector, denoted as $\mathbf{d}_3$ is defined by

$$\mathbf{d}_3(s, t) = \frac{\partial \mathbf{x}(s, t)}{\partial s}. \tag{1}$$

In the standard way the local configuration of the rod is described in terms of the orthonormal *director basis* $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ where $\{\mathbf{d}_1(s, t), \mathbf{d}_2(s, t)\}$ are two differentiable functions spanning the plane normal to the rod such that $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ forms, for each value of the arc-length, a right-handed orthonormal triad with

$$\mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{d}_3,\ \ \mathbf{d}_2 \times \mathbf{d}_3 = \mathbf{d}_1,\ \ \mathbf{d}_3 \times \mathbf{d}_1 = \mathbf{d}_2,\ \ \mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}. \tag{2}$$

As a function of arc-length the directors evolve along the space curve according to

$$\frac{\partial \mathbf{d}_1}{\partial s} = \kappa_3 \mathbf{d}_2 - \kappa_2 \mathbf{d}_3, \tag{3a}$$

$$\frac{\partial \mathbf{d}_2}{\partial s} = \kappa_1 \mathbf{d}_3 - \kappa_3 \mathbf{d}_1, \tag{3b}$$

$$\frac{\partial \mathbf{d}_3}{\partial s} = \kappa_2 \mathbf{d}_1 - \kappa_1 \mathbf{d}_2, \tag{3c}$$

where the $\kappa_i = \kappa_i(s, t)$, $i = 1, 2, 3$ are the components of the *curvature vector*. This system of equations is essentially the director basis analogue of the Frenet equations. For a given value of s the time dependence of the $\mathbf{d}_i$ is governed by an analogous relationship

$$\frac{\partial \mathbf{d}_1}{\partial t} = \omega_3 \mathbf{d}_2 - \omega_2 \mathbf{d}_3, \tag{4}$$

$$\frac{\partial \mathbf{d}_2}{\partial t} = \omega_1 \mathbf{d}_3 - \omega_3 \mathbf{d}_1, \tag{5}$$

$$\frac{\partial \mathbf{d}_3}{\partial t} = \omega_2 \mathbf{d}_1 - \omega_1 \mathbf{d}_2, \tag{6}$$

where $\omega_i = \omega_i(s, t)$, $i = 1, 2, 3$ are the components of the *spin vector*. Note that this form of evolution equation guarantees that the $\mathbf{d}_i$ conserve length since $\partial \mathbf{d}_i \cdot \mathbf{d}_i/\partial t = 2\mathbf{d}_i \cdot \partial \mathbf{d}_i/\partial t = 0$. In the case of time-dependent systems the derivatives of the κ_i with respect to t, and the ω_i with respect to s, are related by the requirement that the cross-derivatives of $\mathbf{d}_i$ are equal, namely

$$\frac{\partial^2 \mathbf{d}_i}{\partial t \partial s} = \frac{\partial^2 \mathbf{d}_i}{\partial s \partial t}. \tag{7}$$

It is important to note that the resulting compatibility relations presuppose that the s and t derivatives commute; thus (7) is only valid for rods whose length do not change in time. We now consider the description of the curve in terms of the *material parameterization*. Denoting the new variable by σ, which labels the position of material points along the curve $\mathbf{x} = \mathbf{x}(\sigma, t)$, the relationship between σ and arc-length is given by

$$s(\sigma, t) = \int_0^\sigma \left(\frac{\partial \mathbf{x}(\sigma', t)}{\partial \sigma'} \cdot \frac{\partial \mathbf{x}(\sigma', t)}{\partial \sigma'} \right)^{1/2} d\sigma', \tag{8}$$

where $s(\sigma, t)$ denotes the arc-length along the curve to the material point σ. A simple example of the two different parameterizations is provided by the case of a circular ring. One can either identify points along the ring by their material angular coordinate, θ, or their arc-length (which measures the distance along the ring from an arbitrary reference point). Clearly $s = R\theta$ where R is the ring radius. In the case where the radius of the ring grows in time, i.e., $R = R(t)$ the material coordinate, θ of any point on the ring remains, of course, unchanged but its arc length will increase. The tangent vector at the point σ on the space curve is defined by

$$\mathbf{T}(\sigma, t) = \frac{\partial \mathbf{x}(\sigma, t)}{\partial \sigma} \tag{9}$$

and, as is well known, this vector may not be of unit length. The unit tangent vector parameterized by σ, which we denote as $\mathbf{d}_3(\sigma, t)$, is given by

$$\mathbf{d}_3(\sigma, t) = \frac{1}{\left\| \frac{\partial \mathbf{x}(\sigma, t)}{\partial \sigma} \right\|} \frac{\partial \mathbf{x}(\sigma, t)}{\partial \sigma}. \tag{10}$$

An alternative way of writing this relationship is through the chain rule; namely

$$\frac{\partial \mathbf{x}(\sigma, t)}{\partial \sigma} = \frac{\partial s}{\partial \sigma} \frac{\partial \mathbf{x}(s, t)}{\partial s} = \lambda(\sigma, t)\mathbf{d}_3(s(\sigma, t), t) = \lambda(\sigma, t)\mathbf{d}_3(\sigma, t), \tag{11}$$

where

$$\lambda(\sigma, t) = \frac{\partial s(\sigma, t)}{\partial \sigma} \tag{12}$$

measures the extensibility of the rod. A rod for which $\lambda = 1$ is traditionally called *inextensible*. Using (11) and (12) we can now write (8) as

$$s(\sigma, t) = \int_0^\sigma |\lambda(\sigma', t)| \, d\sigma'. \tag{13}$$

As with arc-length parameterization, one may form an orthonormal triad of directors in the material parameterization, $\{\mathbf{d}_1(\sigma, t), \mathbf{d}_2(\sigma, t), \mathbf{d}_3(\sigma, t)\}$ which, as a function of σ, evolve along the space curve according to

$$\frac{\partial \mathbf{d}_1}{\partial \sigma} = \tilde{\kappa}_3 \mathbf{d}_2 - \tilde{\kappa}_2 \mathbf{d}_3, \tag{14a}$$

$$\frac{\partial \mathbf{d}_2}{\partial \sigma} = \tilde{\kappa}_1 \mathbf{d}_3 - \tilde{\kappa}_1 \mathbf{d}_1, \tag{14b}$$

$$\frac{\partial \mathbf{d}_3}{\partial \sigma} = \tilde{\kappa}_2 \mathbf{d}_1 - \tilde{\kappa}_1 \mathbf{d}_2. \tag{14c}$$

The $\tilde{\kappa}_i$ can no longer be directly interpreted as the differential geometric curvatures of the space curve; but by comparison with (3) they are seen to be simply related through

$$\tilde{\kappa}_i(\sigma, t) = \lambda\, \kappa_i(s(\sigma, t), t). \tag{15}$$

A spin system for the material director basis can also be written, i.e.,

$$\frac{\partial \mathbf{d}_1}{\partial t} = \Omega_3 \mathbf{d}_2 - \Omega_2 \mathbf{d}_3, \tag{16a}$$

$$\frac{\partial \mathbf{d}_2}{\partial t} = \Omega_1 \mathbf{d}_3 - \Omega_3 \mathbf{d}_1, \tag{16b}$$

$$\frac{\partial \mathbf{d}_3}{\partial t} = \Omega_2 \mathbf{d}_1 - \Omega_1 \mathbf{d}_2, \tag{16c}$$

where the $\Omega_i = \Omega_i(\sigma, t)$ are analogous, but not simply related, to the ω_i in the arc-length spin system (4). One may, in fact, show that

$$\Omega_i(\sigma, t) = \dot{s}(\sigma, t)\kappa_i(s(\sigma, t), t) + \omega_i(s(\sigma, t), t), \quad i = 1, 2, 3, \tag{17}$$

where $\dot{s} = ds/dt$ and our notation $\kappa_i(s(\sigma, t), t)$ indicates that the (arc-length based) curvature κ_i is evaluated at the arc length s corresponding to the material point σ, at the given time t (and similarly for the $\omega_i(s(\sigma, t), t)$). Compatibility between the systems (14) and (16) yields the relations

$$\frac{\partial \Omega_1}{\partial \sigma} - \frac{\partial \tilde{\kappa}_1}{\partial t} = \Omega_3 \tilde{\kappa}_2 - \Omega_2 \tilde{\kappa}_3, \tag{18a}$$

$$\frac{\partial \Omega_2}{\partial \sigma} - \frac{\partial \tilde{\kappa}_2}{\partial t} = \Omega_1 \tilde{\kappa}_3 - \Omega_3 \tilde{\kappa}_1, \tag{18b}$$

$$\frac{\partial \Omega_3}{\partial \sigma} - \frac{\partial \tilde{\kappa}_3}{\partial t} = \Omega_2 \tilde{\kappa}_1 - \Omega_1 \tilde{\kappa}_2, \tag{18c}$$

but unlike the analogous system of relations that can be derived in arc-length parameterization these equations are valid for extensible rods.

2.2. Kirchhoff Equations for an Extensible Rod in Three Dimensions

For a thin rod of arbitrary length, parameterized by the material coordinate σ, the Kirchhoff equations in dimensionless variables are [1, 2, 18]

$$\frac{\partial \mathbf{F}(\sigma, t)}{\partial \sigma} = \frac{\partial^2 \mathbf{x}(\sigma, t)}{\partial t^2}, \tag{19a}$$

$$\frac{\partial \mathbf{M}(\sigma, t)}{\partial \sigma} + \frac{\partial \mathbf{x}}{\partial \sigma} \times \mathbf{F} = \dot{\Pi}(\sigma, t). \tag{19b}$$

In the first of these equation, which represents the balance of linear momentum and which we shall refer to as the Newton's equation, $\mathbf{F}$ is the contact force acting on the cross section of the rod at the material point σ. In the second equation, which represents the balance of moments and which we shall refer to as the moment equation, **M** is the resultant moment and $\dot{\Pi}(\sigma, t)$ the angular momentum. For a rod of circular cross sections this is given by

$$\dot{\Pi}(\sigma, t) = \mathbf{d}_1 \times \frac{\partial^2 \mathbf{d}_1}{\partial t^2} + \mathbf{d}_2 \times \frac{\partial^2 \mathbf{d}_2}{\partial t^2}. \tag{20}$$

The system is closed by the constitutive relations of linear elasticity, namely

$$\mathbf{M}(\sigma, t) = \tilde{\kappa}_1 \mathbf{d}_1 + \tilde{\kappa}_2 \mathbf{d}_2 + \Gamma \tilde{\kappa}_3 \mathbf{d}_3 \tag{21}$$

and the director twist (14) and spin (16) equations. In (21) $\Gamma = 2\mu/E$, namely the ratio of the elastic moduli. The Newton's equation is usually solved by first differentiating through with respect to σ to give

$$\frac{\partial^2 \mathbf{F}(\sigma, t)}{\partial \sigma^2} = \frac{\partial^2}{\partial t^2} \left(\lambda \mathbf{d}_3(\sigma, t) \right), \tag{22}$$

where we have used (11). Evaluating the right hand side using (16) gives

$$\begin{aligned} \mathbf{F}_{\sigma\sigma} = \lambda \mathbf{d}_1 \left(\Omega_{2,t} + 2\frac{\lambda_t}{\lambda}\Omega_2 + \Omega_1\Omega_3 \right) + \lambda \mathbf{d}_2 \left(-\Omega_{1,t} - 2\frac{\lambda_t}{\lambda}\Omega_1 + \Omega_2\Omega_3 \right) \\ + \lambda \mathbf{d}_3 \left(\frac{\lambda_{tt}}{\lambda} - \left(\Omega_1^2 + \Omega_2^2 \right) \right), \end{aligned} \tag{23}$$

where it is understood that all variables have σ as their argument. Setting

$$\mathbf{F} = f_1 \mathbf{d}_1 + f_2 \mathbf{d}_2 + f_3 \mathbf{d}_3 \tag{24}$$

and using (14) the left hand of (23) can be computed; and on equating terms associated with each director one obtains a system of three nontrivial partial differential equations for the three components of **F**.

$$\begin{aligned} & f_{1,\sigma\sigma} - f_1\left(\tilde{\kappa}_2^2 + \tilde{\kappa}_3^2\right) + f_2(\tilde{\kappa}_1\tilde{\kappa}_2 - \tilde{\kappa}_{3,\sigma}) + f_3(\tilde{\kappa}_1\tilde{\kappa}_3 + \tilde{\kappa}_{2,\sigma}) - 2\tilde{\kappa}_3 f_{2,\sigma} + 2\tilde{\kappa}_2 f_{3,\sigma} \\ & \quad = \lambda(\Omega_1\Omega_3 + \Omega_{2,t} + 2\frac{\lambda_t}{\lambda}\Omega_1), \end{aligned} \tag{25a}$$

$$\begin{aligned} & f_{2,\sigma\sigma} - f_2\left(\tilde{\kappa}_2^2 + \tilde{\kappa}_3^2\right) + f_3(\tilde{\kappa}_2\tilde{\kappa}_3 - \tilde{\kappa}_{1,\sigma}) + f_1(\tilde{\kappa}_1\tilde{\kappa}_2 + \tilde{\kappa}_{3,\sigma}) - 2\tilde{\kappa}_1 f_{3,\sigma} + 2\tilde{\kappa}_3 f_{1,\sigma} \\ & \quad = \lambda\left(\Omega_2\Omega_3 - \Omega_{1,t} - 2\frac{\lambda_t}{\lambda}\Omega_2 \right), \end{aligned} \tag{25b}$$

$$\begin{aligned} & f_{3,\sigma\sigma} - f_3\left(\tilde{\kappa}_2^2 + \tilde{\kappa}_3^2\right) + f_1(\tilde{\kappa}_1\tilde{\kappa}_3 - \tilde{\kappa}_{2,\sigma}) + f_2(\tilde{\kappa}_2\tilde{\kappa}_3 + \tilde{\kappa}_{1,\sigma}) - 2\tilde{\kappa}_2 f_{1,\sigma} + 2\tilde{\kappa}_1 f_{2,\sigma} \\ & \quad = \lambda\left(\frac{\lambda_{tt}}{\lambda} - {\Omega_1}^2 - {\Omega_2}^2 \right). \end{aligned} \tag{25c}$$

The force component f_3 has a special meaning, namely it corresponds to the *tension* in the rod. For an extensible rod one has the additional constitutive relation

$$f_3(\sigma, t) = \lambda(\sigma, t) - 1. \tag{26}$$

If the rod is inextensible, $\lambda = 1$. However, this does not mean that one now sets $f_3 = 0$! The inextensible case is a highly singular limit: there is no constitutive relation in this case and the tension in the rod is determined by the differential equation governing f_3. We now turn to the form of the moment equation to be coupled with (23). Using the constitutive relations (21) and the director equations (14) one can derive the following system of equations:

$$\tilde{\kappa}_{1,\sigma} + (\Gamma - 1)\tilde{\kappa}_2\tilde{\kappa}_3 - \lambda f_2 = \Omega_2\Omega_3 + \Omega_{1,t}, \tag{27a}$$

$$\tilde{\kappa}_{2,\sigma} - (\Gamma - 1)\tilde{\kappa}_1\tilde{\kappa}_3 + \lambda f_1 = -\Omega_1\Omega_3 + \Omega_{2,t}, \tag{27b}$$

$$\Gamma\tilde{\kappa}_{3,\sigma} = 2\Omega_{3,t}. \tag{27c}$$

We note that differentiating the last of these equations with respect to σ, and using the compatibility conditions (18) yields an inhomogeneous wave equation for $\tilde{\kappa}_3$ of the form

$$\frac{\Gamma}{2}\tilde{\kappa}_{3,\sigma\sigma} = \tilde{\kappa}_{3,tt} + (\Omega_2\tilde{\kappa}_1 - \Omega_1\tilde{\kappa}_2)_t. \tag{28}$$

3. Amplitude Equations for an Extensible Rod in $\mathbb{R}^3$

A major result of our earlier studies was the use of weakly nonlinear analysis to reduce the Kirchhoff equations (a system of six coupled partial differential equations second order in both space and time) to a system of nonlinear amplitude equations [16]. For the case of a straight, inextensible, twisted rod (in three dimensions) with circular cross-section, the simplest version of these equations takes the form

$$\left(\frac{1+P^2}{P^2}\right)\frac{\partial^2 A}{\partial T^2} - \frac{\partial^2 A}{\partial S^2} = A\left(2P\Gamma\Psi + P\Gamma\frac{\partial B}{\partial S} - 2P^2\Gamma|A|^2\right), \tag{29a}$$

$$\frac{2}{\Gamma}\frac{\partial^2 B}{\partial T^2} - \frac{\partial^2 B}{\partial S^2} = -2P\frac{\partial |A|^2}{\partial S}, \tag{29b}$$

where $A(S, T)$ is the amplitude and $B(S, T)$ the twist density of the solution beyond the bifurcation threshold, and P is an externally prescribed tension. The independent variables S and T are the stretched space and time variables where

$$S = \epsilon s, \qquad T = \epsilon t,$$

with ϵ measuring the distance of the bifurcation parameter from threshold, namely $\gamma - \gamma_c = \Psi\epsilon^2$, where Ψ is a number of $O(1)$. A subsequent modification of these equations included coupling to a third equation for the variable $C(S, T)$ corresponding to the modal tension [17]. For the case of a straight, inextensible, twisted rod with circular cross-section, the amplitude equations then take the form

$$\left(\frac{1+P^2}{P^2}\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial S^2}\right)A = A\left(2P\Gamma\Psi - C + P\Gamma\frac{\partial}{\partial S}B - 2P^2\Gamma|A|^2\right), \tag{30a}$$

$$-\frac{\partial^2}{\partial S^2}C = 2\left(\frac{\partial^2}{\partial T^2} + P^2\frac{\partial^2}{\partial S^2}\right)|A|^2, \tag{30b}$$

$$\left(\frac{2}{\Gamma}\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial S^2}\right)B = -2P\frac{\partial}{\partial S}|A|^2. \tag{30c}$$

It should be noted that the additional equation for the tension mode C arises as a solubility condition at $O(\epsilon^4)$ in the multiple-scales expansion. This enlarged system of equations, even without the inclusion of extensibility, has yet to be subjected to detailed study. The original system of amplitude equations given above (29), even without the effects of extensibility and coupling of the tension mode, has proved useful for a number of applications. It has also been the subject of more detailed analytical and numerical studies [24] and, more recently, was the subject of a stability analysis [20, 21].

3.1. The Inertial Case

Deriving amplitude equations for the case of rods is a substantial (ad)venture in its own right. The analysis follows exactly the same formalism introduced in [15, 16] and, as with the previously derived amplitude equations, involves the same scalings of the spatial and temporal variables. Accordingly we do not give the details here and just give the final result. To include the effect of extensibility in the most general way we write the constitutive relation for the tension in the form

$$\lambda = 1 + \chi f_3. \tag{31}$$

In this formulation the parameter χ can take two values:
(i) $\chi = 0$ corresponding to an inextensible rod, and
(ii) $\chi = 1$ corresponding to the extensible case.

Carrying through the amplitude equation analysis for a straight twisted rod with circular cross-section yields

$$\left(\frac{(1+(1+\chi)P^2)}{P^2}\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial S^2}\right)A = A(2P\Gamma\Psi\sqrt{1+\chi P^2} - \kappa(1+2\chi P^2)C \\ + P\Gamma\sqrt{1+\chi P^2}\frac{\partial}{\partial S}B - 2P^2(\Gamma + \chi P^2(1+\Gamma))|A|^2), \tag{32a}$$

$$\left(\chi\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial S^2}\right)C = 2\left(\frac{\partial^2}{\partial T^2} + P^2\frac{\partial^2}{\partial S^2}\right)|A|^2, \tag{32b}$$

$$\left(\frac{2}{\Gamma}\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial S^2}\right)B = -2P\sqrt{1+\chi P^2}\frac{\partial}{\partial S}|A|^2. \tag{32c}$$

The symbol κ (not to be confused with the curvature components κ_i) on the right hand side of the first of these equations is introduced for convenience: it takes on either the value $\kappa = 1$ corresponding to the coupling of the tension C to the amplitude A, or the value $\kappa = 0$ which results in elimination of this coupling. Thus the case $\chi = 0, \kappa = 1$ reduces the general system to (30), and the case $\chi = 0, \kappa = 0$ reduces it to (29). The effect of extensibility, i.e., $\chi = 1$ is to modify the various coefficients, in effect strengthening the nonlinearity, $A|A|^2$, in the amplitude equation for A. As will be seen shortly this is especially significant in the dissipative case.

3.2. The Dissipative Case

If the filament is submerged in a viscous environment it may be possible to neglect inertial terms and represent the dissipative effects in a *local* approximation. This leads to Kirchhoff equations of the form:

$$\frac{\partial \mathbf{F}(\sigma,t)}{\partial \sigma} = \Lambda \frac{\partial \mathbf{x}(\sigma,t)}{\partial t}, \tag{33a}$$

$$\frac{\partial \mathbf{M}(\sigma,t)}{\partial \sigma} + \frac{\partial \mathbf{x}}{\partial \sigma} \times \mathbf{F} = \dot{\Pi}(\sigma,t), \tag{33b}$$

where Λ denotes viscosity of the environment. It should of course be noted that the dynamics of a slender body in a fluid is much more complicated, and a more complete picture requires the introduction of nonlocal effects [25]. Given the above local approximation the amplitude equation formalism can be carried through as before. However, an analysis of the dispersion relations shows that the scalings are now different from the inertial case, namely

$$S = \epsilon s, \qquad T = \epsilon^2 t.$$

The simplest case to consider is that of an inextensible, twisted rod with circular cross-section. Neglecting the tension mode gives

$$\frac{\Lambda}{P^2}\frac{\partial A}{\partial T} - \frac{\partial^2 A}{\partial S^2} = A\left(2P\Gamma\Psi + P\Gamma\frac{\partial B}{\partial S} - 2P^2\Gamma|A|^2\right), \tag{34a}$$

$$-\frac{\partial^2 B}{\partial S^2} = -2P\frac{\partial |A|^2}{\partial S}, \tag{34b}$$

which can be compared directly with (29). The important thing to note in this case is that the twist density equation can be integrated once, and on substituting the resulting form of $\partial B/\partial S$ into the equation for A there is an exact cancellation of the nonlinear terms and one is left with

$$\frac{\Lambda}{P^2}\frac{\partial A}{\partial T} - \frac{\partial^2 A}{\partial S^2} = 2\Gamma P\Psi A.$$

Thus both the amplitude and twist (which is slaved to the amplitude) decay diffusively – there is no possibility of a localized twist wave propagating down the filament. If one now repeats the analysis including the effects of extension and the tension mode the amplitude equations take the form:

$$\left(\frac{\Lambda(1+\chi)P^2}{P^2}\frac{\partial}{\partial T} - \frac{\partial^2}{\partial S^2}\right)A = A(2P\Gamma\Psi\sqrt{1+\chi P^2} - \kappa(1+2\chi P^2)C$$
$$+ P\Gamma\sqrt{1+\chi P^2}\frac{\partial}{\partial S}B - 2P^2(\Gamma + \chi P^2(1+\Gamma))|A|^2), \tag{35a}$$

$$\left(\Lambda\chi\frac{\partial}{\partial T} - \frac{\partial^2}{\partial S^2}\right)C = 2\left(\Lambda\frac{\partial}{\partial T} + P^2\frac{\partial^2}{\partial S^2}\right)|A|^2, \tag{35b}$$

$$-\frac{\partial^2}{\partial S^2}B = -2P\sqrt{1+\chi P^2}\frac{\partial}{\partial S}|A|^2. \tag{35c}$$

The equation for the twist density B may be again integrated once, and on using this in the amplitude equation for A one finds that this equation becomes

$$\left(\frac{\Lambda(1+\chi)P^2}{P^2}\frac{\partial}{\partial T} - \frac{\partial^2}{\partial S^2}\right)A = A(2P\Gamma\Psi\sqrt{1+\chi P^2} - \kappa(1+2\chi P^2)C - 2\chi P^4|A|^2). \tag{36}$$

The crucial point to observe here is that in the extensible case, i.e. $\chi = 1$, the nonlinear term $A|A|^2$ is preserved leading to the possibility of nonlinear wave solutions, even in the absence of coupling to the tension.

4. Solutions to the Amplitude Equations

In this section, localized solutions to the amplitude Equations (32) are found. We consider three cases:
(i) $\chi = \kappa = 1$ (extensible with tension mode),
(ii) $\chi = 0,\ \kappa = 1$ (inextensible with tension mode),
(iii) $\chi = \kappa = 0$ (inextensible without a tension mode).
In the rest of this article, the lower case letters s and t will be used for the independent variables in the amplitude Equations (32) instead of S and T.

4.1. General Form of the Solutions

We consider system (32) and look for solutions of the form

$$A = a(\xi)e^{i\omega t}, \quad B = b(\xi), \quad C = c(\xi), \tag{37}$$

where $\xi = s - v\,t$. The functions a, b, and c satisfy a system of three ODEs, one complex equation for a, and real equations for b and c. The latter two equations can be directly integrated so that b' and c can be expressed in terms of a

$$b' = -2P\Gamma\frac{\sqrt{1+\chi P^2}}{2v^2-\Gamma}\,|a|^2 + K_1, \tag{38}$$

$$c = 2\frac{v^2+P^2}{\chi\, v^2-1}|a|^2 + K_2\xi + K_3, \tag{39}$$

where K_1, K_2, and K_3 are constants of integration. The constant K_2 vanishes identically since the tension mode cannot grow linearly in ξ. The complex equation for a takes the form

$$\begin{aligned}\left(\frac{v^2(1+(1+\chi)P^2)}{P^2} - 1\right)a'' = {} & 2iv\omega\frac{1+(1+\chi)P^2}{P^2}a' \\ & + a\left(P\Gamma\sqrt{1+\chi P^2}(2\Psi + K_1) - \kappa(1+2\chi P^2)K_3\right. \\ & \left. + \frac{1+(1+\chi)P^2}{P^2}\omega^2 + \Omega\,|a|^2\right),\end{aligned} \tag{40}$$

where

$$\Omega = \begin{cases} 2\dfrac{(1+P^2)(P^2(2v^4(1+\Gamma)+v^2(2-3\Gamma)-\Gamma)+v^2(2v^2-\Gamma))}{(v^2-1)(\Gamma-2v^2)} & \text{if } \kappa = \chi = 1, \\ 2\,\dfrac{P^2(2\,v^2\Gamma + 2\,v^2P^2 + 2\,P^2v^2\Gamma - P^2\Gamma)}{2\,v^2-\Gamma} & \text{if } \kappa = 1,\ \chi = 0, \\ 4\,\dfrac{P^2v^2\Gamma}{2\,v^2-\Gamma} & \text{if } \kappa = \chi = 0. \end{cases} \tag{41}$$

The localized solutions for an equation of the same form as (40) (with different coefficients) were found and classified in [24] (see also [26] for a similar reduction in a different context). To solve for a we introduce its polar decomposition

$$a = R(\xi)\, e^{i\theta(\xi)}, \tag{42}$$

and the equation can be separated into two real equations, each of which can be integrated once. It is then possible to find and classify pulse, hole, and front types of solutions for a. Here, we only consider pulse solutions.

4.2. Travelling Pulses

The pulse solutions take the form

$$a_0 = \alpha \operatorname{sech}(\beta\xi)\, \exp(i\sigma\xi), \quad b_0 = \delta \tanh(\beta\xi), \quad c_0 = \gamma \operatorname{sech}^2(\beta\xi), \tag{43}$$

where

$$\alpha^2 = \frac{(2\,v^2 - \Gamma)(\chi\, v^2 - 1)(v^2 + P^2(v^2(1+\chi) - 1))\,\beta^2}{P^2(1 + \chi\, P^2)(2\,v^4\,(1 + (1+\Gamma)\chi\, P^2) + v^2\,((2 - \Gamma(\chi+2))P^2 - \Gamma) - P^2\Gamma)} \tag{44}$$

in the case $\kappa = 1$,

$$\alpha^2 = \frac{(\Gamma - 2\,v^2)(v^2 + P^2(v^2 - 1))\,\beta^2}{2\,P^4 v^2 \Gamma}, \tag{45}$$

in the case $\kappa = \chi = 0$ and

$$\beta^2 = \frac{2\,P^3\,\Psi\,\sqrt{1 + \chi\, P^2}}{v^2 + P^2(v^2(1+\chi) - 1)} - \omega^2 \frac{P^2(1 + P^2(1+\chi))}{(v^2 + P^2(v^2(1+\chi) - 1))^2}, \tag{46}$$

$$\delta = 2\,\frac{\alpha^2\, P\sqrt{1 + \chi\, P^2}\,\Gamma}{\beta\,(\Gamma - 2\,v^2)}, \tag{47}$$

$$\gamma = 2\frac{\alpha^2\,(v^2 + P^2)}{\chi\, v^2 - 1}, \tag{48}$$

$$\sigma = \frac{\omega\, v(1 + P^2(1+\chi))}{v^2 + P^2(v^2(1+\chi) - 1)}. \tag{49}$$

A necessary condition for the existence of a pulse is that α and β must be real. Since we consider elastic rods under tension, the parameter P is positive, and the parameter Γ, characterizing the elastic property of the rod lies between 2/3 (incompressible case) and 1 (hyperelastic case). The parameter Ψ is a measure of the distance from the bifurcation point and is a positive (postbifurcation) or negative (prebifurcation) real number close to 0. If $\Psi < 0$, there exists an interval of values of ω for which the right-hand-sides of (44) (or (45)) and (46) are both positive if and only if

$$v^2 + P^2(v^2(1+\chi) - 1) < 0, \tag{50}$$

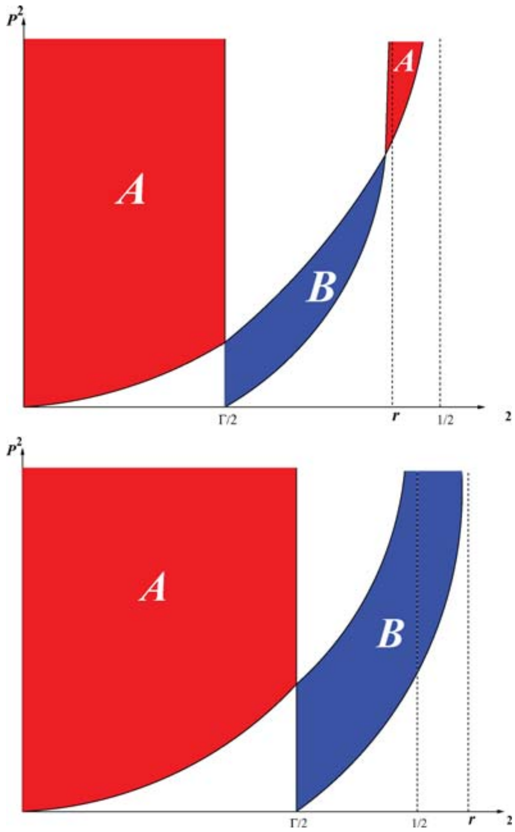

Figure 1. Regions of existence for the pulse solutions (43) in the case $\kappa = \chi = 1$. The regions labelled by A correspond to the case $\Psi < 0$ and the regions labelled by B to $\Psi > 0$. The top figure corresponds to the case $2/3 < \Gamma < 3/4$ and the bottom one corresponds to $3/4 < \Gamma < 1$. On both figures $r = -2 + 3\,\Gamma + \sqrt{4 - 4\,\Gamma + 17\,\Gamma^2}/4 + 4\,\Gamma$. $r > 1/2$ for $\Gamma > 3/4$.

for $\kappa = 0$ or 1 and

$$((2 + (2 + 2\,\Gamma)\chi\, P^2)v^4 + ((2 - \Gamma(\chi + 2))P^2 - \Gamma)v^2 - P^2\Gamma)(\Gamma - 2\,v^2) < 0, \tag{51}$$

for $\kappa = 1$ and

$$(\Gamma - 2\,v^2) < 0 \tag{52}$$

for $\kappa = \chi = 0$. When $\Psi > 0$, the inequalities (50), (51), and (52) are reversed. For a fixed value of Γ, the condition (50) with condition (51) or (52) define a region of the (v^2, P^2) space. The regions of existence of pulse solutions in the different cases are illustrated in Figures 1–3.

5. Stability Analysis

In this section, we study the stability of the pulse solutions (43) in the extensible case (i.e., when $\chi = 1$). We use two notions of stability: linear and orbital stability. In the case of linear stability, one considers

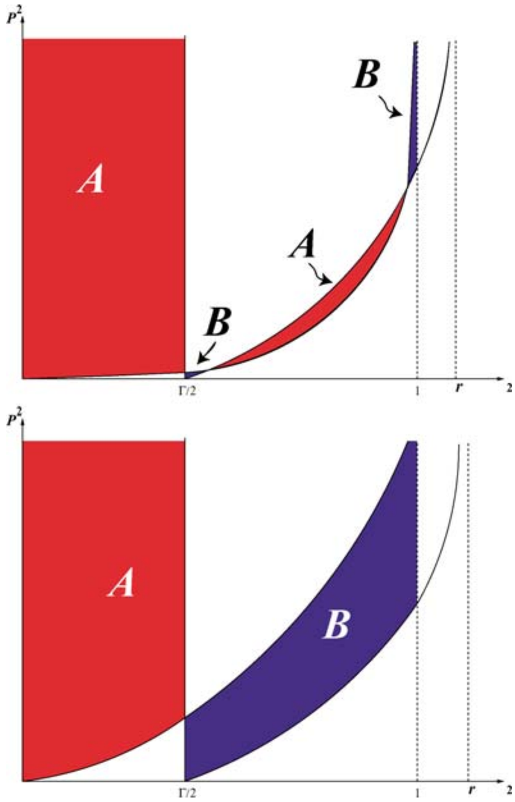

Figure 2. Regions of existence for the pulse solutions (43) in the case $\kappa = 1$ and $\chi = 0$. The regions labelled by A correspond to the case $\Psi < 0$ and the regions labelled by B to $\Psi > 0$. The top figure corresponds to the case $2/3 < \Gamma < 12 - 8\sqrt{2}$ and the bottom one corresponds to $12 - 8\sqrt{2} < \Gamma < 1$. The asymptote is given by $r = \Gamma/2(1 - \Gamma)$. $r > 1$ for $\Gamma > 2/3$.

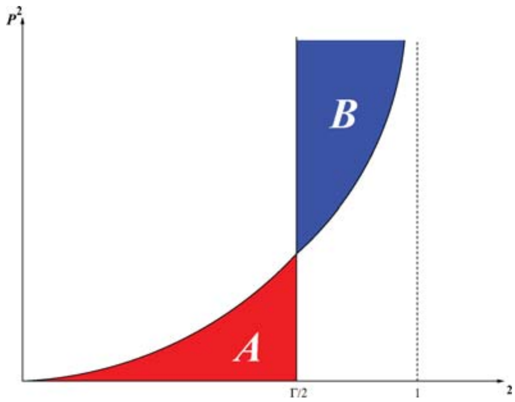

Figure 3. Regions of existence for the pulse solutions (43) in the case $\kappa = \chi = 0$. The region labelled by A corresponds to the case $\Psi < 0$ and the region labelled by B to $\Psi > 0$.

a localized solution and studies the properties of the linear operator obtained from the linearization of (32) about the solution. In this context, a localized solution is *linearly stable* if no element of the spectrum of the linear operator has a positive real part. The notion of orbital stability is defined in [27, 28] by Grillakis et al., and in this article we follow their methodology. To define orbital stability, one considers a family of localized solutions such as (43) and, roughly speaking, a localized solution is *orbitally stable* if a small change in its initial conditions gives rise to a solution that remains close to one of the members of the family. Orbital stability implies linear stability but it is generally easier to prove the latter. For $\Psi > 0$ the pulse solutions are always unstable. To show this, we linearize the system (32) about the asymptotic state $A = C = 0$ and B constant. Amplitude perturbations of the form $\exp[i\, ks + \Omega\, t]$ have a dispersion relation given by

$$\frac{1 + 2P^2}{P^2}\Omega^2 = 2\, P \Gamma\, \Psi \sqrt{1 + P^2} - k^2. \tag{53}$$

This indicates that when $\Psi > 0$, Fourier modes with wave number k such that $k^2 < 2\, P\, \Gamma\, \Psi \sqrt{1 + P^2}$ experience growth. As a consequence, when $\Psi > 0$, the asymptotic state of the pulse solution is linearly unstable. In terms of linear operators, the continuous spectrum of the linearization about the pulse solution intersects the open right half side of the complex plane. When $\Psi < 0$ the situation is more subtle. Although (53) indicates that the asymptotic state of the solutions is now stable, this does not necessarily imply that the pulses themselves are stable. However, we can take advantage of the conservative character of (32) to study the stability of the pulse solutions using the method of Grillakis et al. Before we proceed with this calculation we give a general overview of this method.

5.1. The General Methods to Study Orbital Stability

The technique used to study the orbital stability of localized solutions is based on an analysis of systems of the form

$$\frac{\partial\, \mathbf{v}}{\partial t} = J\, \delta E(\mathbf{v}) \tag{54}$$

where $\mathbf{v}(s, t) \in X$ is an n-dimensional vector which depends on the space coordinate s and on time t, X is a real Hilbert space, J is an invertible $n \times n$ skew-symmetric matrix, E is a functional of $\mathbf{v}$, and $\delta\, E$ is the Fréchet derivative of E with respect to its argument. We assume that there exists a one-parameter group T of unitary transformations on X which commute with J and leave the Hamiltonian system invariant. Furthermore, we assume that there exists a one-parameter family of solutions $\mathbf{u}(s, t)$ to (54) which can be written

$$\mathbf{u}(s, t) = T(rt)\, \mathbf{u}_r(s), \tag{55}$$

where $\mathbf{u}_r(s)$ depends only on space and is parameterized by the one-dimensional parameter r. In order to study the linear stability of $\mathbf{u}(s, t)$, one first linearizes (54) about this one-parameter family of solutions. The linearized system reads

$$\frac{\partial \mathbf{w}}{\partial t} = J\, H_r\, \mathbf{w}, \tag{56}$$

where the perturbation to $\mathbf{u}(s, t)$ is $T(rt)\, \mathbf{w}(s, t)$, H_r is a self-adjoint operator given by

$$H_r = \delta^2 E(\mathbf{u}_r) - r\, \delta^2 Q_r(\mathbf{u}_r),$$

and Q_r is the conserved quantity associated with the invariance T and obtained from T by means of an infinite-dimensional version of Noether's Theorem [27–29]. The solution (55) is then a critical point of the functional

$$E(\mathbf{u}_r) - r\, Q_r(\mathbf{u}_r). \tag{57}$$

This can be intuitively understood as an infinite-dimensional version of the Lagrange multiplier method. Indeed, one proves stability by showing that the solution is a minimizer of the energy functional under the constraint that Q_r remains constant. The parameter r plays the role of the Lagrange multiplier and the second derivative of (57) can have a negative eigenvalue in the direction perpendicular to the level sets of Q_r. The method takes advantage of the fact that H_r is self-adjoint and the spectral structure of H_r is used to obtain spectral properties of the linearization of the flow given by $J\, H_r$. More precisely, assuming that H_r has exactly one negative eigenvalue, the convexity requirement

$$\frac{d^2}{dr^2}\, d(r) > 0,$$

where the scalar function $d(r)$ is given by

$$d(r) = E(\mathbf{u}_r) - r\, Q_r(\mathbf{u}_r),$$

is a necessary and sufficient condition for the stability of (55), provided the continuous spectrum of H_r is positive and bounded away from the origin. This method has been generalized to the case where there is more than one symmetry. These symmetries are represented by unitary transformations preserving the Hamiltonian structure [28]. In this case, the parallel with Lagrange multipliers method still holds. If there are m symmetries, T_i, $i = 1, \ldots, m$, then the parameter r is replaced by the m-dimensional vector $\mathbf{r}$ and the family of solutions to be studied contains n parameters and has the form

$$\mathbf{u}(s,t) = T_1(r_1 t) \circ T_2(r_2 t) \circ T_3(r_3 t) \circ \cdots \circ T_m(r_m t)\, \mathbf{u}_{\mathbf{r}}(s). \tag{58}$$

The linearized system now reads

$$\frac{\partial \mathbf{w}}{\partial t} = J\, H_{\mathbf{r}}\, \mathbf{w}, \tag{59}$$

where $H_{\mathbf{r}}$ is the self-adjoint operator given by

$$H_{\mathbf{r}} = \delta^2 E(\mathbf{u}_{\mathbf{r}}) - \sum_{i=1}^{m} r_i\, \delta^2 Q_{r_i}(\mathbf{u}_{\mathbf{r}}). \tag{60}$$

Let $d(\mathbf{r})$ be the m-dimensional vector-valued function

$$d(\mathbf{r}) = E(\mathbf{u}_{\mathbf{r}}) - \sum_{i=1}^{m} r_i\, Q_{r_i}(\mathbf{u}_{\mathbf{r}}),$$

and let $d''(\mathbf{r})$ denote the Hessian of d, where $d''(\mathbf{r})$ is assumed to be nonsingular, and $p(d)$ be the number of positive eigenvalue of d''. If the dimension of the negative spectrum of $H_{\mathbf{r}}$, denoted $n(H_{\mathbf{r}})$, is finite, the following conditions apply provided the continuous spectrum of $H_{\mathbf{r}}$ is positive and bounded away from the origin:

- If $n(H_{\mathbf{r}}) = p(d)$, then the solution is orbitally stable.
- If $n(H_{\mathbf{r}}) - p(d) > 0$ is odd, then the solution is orbitally unstable.

In what follows, we apply the generalized method with two symmetries.

5.2. Hamiltonian Form of the Amplitude Equations

The first step to apply the method is to identify combinations of the variables and an energy functional so that system (32) with extensibility can be written in the Hamiltonian form (54). To do so, we introduce the variables $\mathbf{v} = (R_1, R_2, U, W, P_1, P_2, B, F)^{\mathrm{T}}$ where

$$F = \int (C - 2|A|^2)\, ds, \tag{61}$$

and the real variables P_1, P_2, R_1, R_2, U, and W are defined through

$$A = P_1 + iP_2, \quad A_t = \frac{P^2}{1 + (1+\kappa)\, P^2}\, (R_1 + iR_2), \quad U = B_t, \quad W = \frac{1 + 2P^2}{2(1 + P^2)} F_t. \tag{62}$$

Then, the system (32) can be written in the form (54), and

$$E(\mathbf{v}) = \int_{-\infty}^{\infty} h\, ds, \tag{63}$$

with

$$\begin{aligned} h(s,t) = {} & \frac{1 + (1+\kappa)\, P^2}{P^2} |A_t|^2 + \frac{1}{2} B_t^2 + |A_s|^2 - 2P\Gamma\Psi\sqrt{1 + \kappa\, P^2}|A|^2 \\ & + (P^2(\Gamma + 2\kappa + \kappa\, P^2(1 + \Gamma)) + \kappa)|A|^4 - P\Gamma\sqrt{1 + \kappa\, P^2}|A|^2 B_s \\ & + \frac{\Gamma}{4} B_s^2 + \kappa\,(1 + 2P^2) \left(\frac{F_t^2 + F_s^2}{4(1 + P^2)} + |A|^2 F_s \right), \end{aligned} \tag{64}$$

and

$$J = \begin{pmatrix} 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{65}$$

Note that in the case $\kappa = 0$, there is no tension mode C and the contributions of the variables F and W vanish identically.

5.3. Conditions of Application

Once the system has been written in Hamiltonian form, there are still some technical conditions that have to be satisfied in order to test for orbital stability. In our case, these amount to showing that the following conditions hold:

1. The pulse solutions have the form given in (58).
2. The self-adjoint operator $H_\mathbf{r}$ defined in (60) has a finite-dimensional negative subspace.

Furthermore, the continuous part of spectrum of the operator $H_\mathbf{r}$ should be positive and bounded away from zero. In our case, the continuous spectrum includes zero. Because of this fact only linear stability (as opposed to orbital stability) results can be obtained.

5.3.1. *Symmetries*

In order to verify the first condition we must show that the two-parameter family of pulse solutions (37) and (43) can be written as (58). There are two unitary transformations that preserve the Hamiltonian structure (63). They are both elements of the symmetry group on (32). We use the following notation: $A(s,t)$, $B(s,t)$, and $C(s,t)$ are solutions of system (32) and the new solutions $\tilde{A}(s,t)$, $\tilde{B}(s,t)$, and $\tilde{C}(s,t)$ are obtained by applying the following symmetries to A, B, and C.

1. Reference frame invariance $T_1(s_0)$:
$$\tilde{A}(s,t) = A(s+s_0,t), \quad \tilde{B}(s,t) = B(s+s_0,t), \quad \tilde{C}(s,t) = C(s+s_0,t), \tag{66}$$
where s_0 is a real arbitrary constant.
2. Gauge invariance $T_2(\theta)$:
$$\tilde{A}(s,t) = A(s,t)\,e^{i\theta}, \quad \tilde{B}(s,t) = B(s,t), \quad \tilde{C}(s,t) = C(s,t), \tag{67}$$
where θ is a real arbitrary constant.

The pulse solution defined in (37) and (43) can then be written in the desired way, that is as a function of s and t whose temporal dependence is generated by the group action:

$$(A(s,t), B(s,t), C(s,t)) = T_1(ct) \circ T_2(\omega t)\,(a_0(s), b_0(s), c_0(s)), \tag{68}$$

where a_0, b_0, and c_0 are defined in (43). The invariance and gauge invariance symmetries T_1 and T_2 of (66) and (67) are associated with two conserved quantities

$$Q_r(\mathbf{v}) = -\int_{-\infty}^{\infty} (2(R_1 P_{1s} + R_2 P_{2s}) + UB_s + WF_s)ds, \tag{69}$$

$$Q_g(\mathbf{v}) = 2\int_{-\infty}^{\infty} (R_2\, P_1 - P_2\, R_1)\, ds. \tag{70}$$

The solution (a_0, b_0, f_0) defined in (43) with f_0 defined in (61) by

$$f_0 = \int (c_0 - 2\,|a_0|^2)\, d\xi \tag{71}$$

is then a critical point for the functional

$$I(\mathbf{v}) = E - v\, Q_r - \omega\, Q_g. \tag{72}$$

This can be verified by showing that

$$\delta I = \begin{pmatrix} 2\left(\frac{P^2}{1+(1+\kappa)\,P^2}\,R_1 + v\,P_{1s} + \omega\,P_2\right) \\ 2\left(\frac{P^2}{1+(1+\kappa)\,P^2}\,R_2 + v\,P_{2s} - \omega P_1\right) \\ U + v\,B_s \\ \frac{2\,\kappa\,(1+P^2)}{1+2\,P^2}\,W + v\,F_s \\ 2(-P_{1ss} + P_1(-2\,P\,\Gamma\,\Psi\,\sqrt{1+\kappa\,P^2} + (P^2(\Gamma + 2\,\kappa + \kappa\,P^2(1+\Gamma)) + \kappa)\big(P_1^2 + P_2^2\big) \\ \quad -P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,B_s + \kappa\,(1+2\,P^2)\,F_s) - v\,R_{1s} - \omega R_2) \\ 2(-P_{2ss} + P_2(-2\,P\,\Gamma\,\Psi\,\sqrt{1+\kappa\,P^2} + (P^2(\Gamma + 2\,\kappa + \kappa\,P^2(1+\Gamma)) + \kappa)\big(P_1^2 + P_2^2\big) \\ \quad -P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,B_s + \kappa\,(1+2\,P^2)\,F_s) - v\,R_{2s} + \omega R_1) \\ -\frac{\Gamma}{2}\,B_{ss} + P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,\big(P_1^2 + P_2^2\big)_s - v\,U_s \\ -\kappa\,(1+2\,P^2)\left(\frac{1}{2\,(1+P^2)}\,F_{ss} + \big(P_1^2 + P_2^2\big)_s\right) - v\,W_s \end{pmatrix} = 0 \tag{73}$$

is equivalent to the equations satisfied by a, b, and f defined in (43) with the equation for a separated in real and imaginary parts and the equation for c being replaced by the equation for f.

5.3.2. *Eigenvalues*

The second condition necessary for the application of the method concerns the eigenvalues of the linearization of (73) around the solution (43). Let $\mathbf{v}_0 = (r_1, r_2, u_0, w_0, p_1, p_2, b_0, f_0)$ be the six-dimensional solution of (73) defined by a_0, b_0, and c_0 of (43) with

$$a_0 = p_1 + i p_2, \quad r_1 = -v\,p_1' - \omega\,p_2, \quad r_2 = -v\,p_2' + \omega\,p_1, \quad f_0 = \int (c_0 - 2\,|a_0|^2)\,ds. \tag{74}$$

Then, we have to show that the linear operator

$$H_{v,\omega} \equiv \delta^2 I(\mathbf{v}_0) = \delta^2 E(\mathbf{v}_0) - v\delta^2 Q_r(\mathbf{v}_0) - \omega\delta^2 Q_g(\mathbf{v}_0) \tag{75}$$

has a finite number of negative eigenvalues and that the continuous spectrum of $H_{v,\omega}$ is positive and bounded away from the origin. We show in Appendix A that there is actually only one negative eigenvalue. However, the continuous spectrum consists of all positive real numbers including zero and thus is not bounded away from the origin. Therefore orbital stability cannot be proved. In such a situation, it is shown in [20] that the criterion for orbital stability still provides a criterion for linear stability. That is

- If $n(H_{\mathbf{r}}) = p(d)$, then the solution is linearly stable, that is the spectrum of the operator arising from linearization of (32) around the solution does not intersect the open right side of the complex plane.
- If $n(H_{\mathbf{r}}) - p(d) > 0$ is odd, the solution is unstable.

5.4. A Linear Stability Criterion

It is now possible to obtain a necessary and sufficient criterion for the linear stability of the pulse solutions. Consider the scalar function of v and ω given by

$$d(v, \omega) = E(\mathbf{v}_0) - v\, Q_r(\mathbf{v}_0) - \omega\, Q_g(\mathbf{v}_0), \tag{76}$$

where E, Q_r, and Q_g are defined in (63), (69), and (70) and $\mathbf{v}_0$ is the pulse solution. Note that d depends on v and ω both explicitly in (76) and implicitly through the dependence of $\mathbf{v}_0$ on v and ω. In the case we are interested in, i.e., the case with two symmetries and $H_{v,\omega}$ having a one-dimensional negative subspace, the criterion given at the end of Section 5.1 can be stated as follows: the pulse is linearly

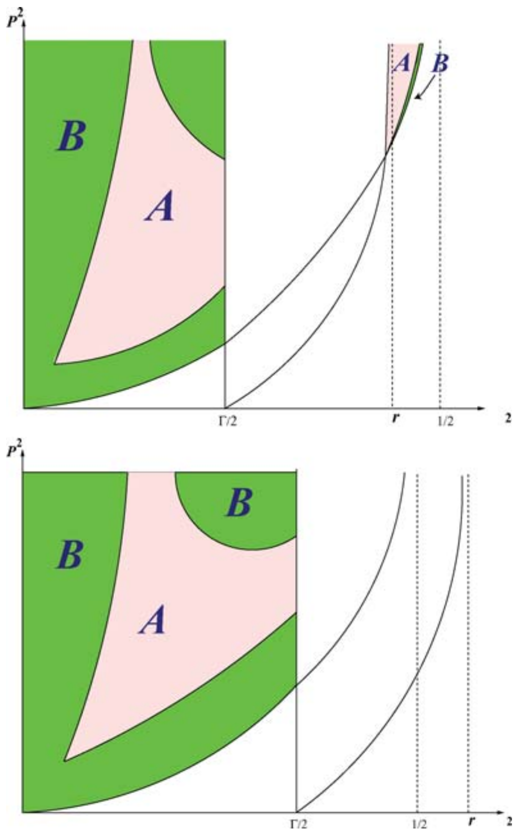

Figure 4. Regions of linear stability for the pulse solutions (43) in the case $\kappa = \chi = 1$ and $\Psi < 0$. The regions labelled by B correspond to the values of P and v for which the pulse is linearly unstable, the regions labelled by A correspond to linear stability. The top figure corresponds to the case $2/3 < \Gamma < 3/4$ and the bottom one corresponds to $3/4 < \Gamma < 1$. On both figures $r = -2 + 3\Gamma + \sqrt{4 - 4\Gamma + 17\Gamma^2}/4 + 4\Gamma$. $r > 1/2$ for $\Gamma > 3/4$. Note that the upper right green region in the bottom figure collapses and disappears for Γ greater than approximatively 0.8124.

stable if and only if

$$d'' = d_{vv}\, d_{\omega\omega} - d_{v\,\omega}^2 < 0. \tag{77}$$

In the case $\kappa = \chi = 1$, the stability condition obtained from (77) is written explicitly in Appendix B for $\omega = 0$ only. The general condition with $\omega \neq 0$ is too cumbersome to be written out explicitly.

6. Conclusions

The stability analysis of particular solutions of mechanical systems such as elastic rods is crucial to understand the true behavior of solutions but is notoriously difficult. In this paper, we looked at the problem of analyzing the stability of pulse solutions in extensible twisted rods under tension. In this regime, a weakly nonlinear analysis can be performed and new amplitude equations have been derived. The analysis of these equations reveals the existence of pulse solutions before and after the bifurcation point. After the bifurcation point, the straight twisted solutions are unstable and, as a consequence, the pulse solutions which connect such states are also unstable. The analysis of solutions before bifurcation reveals a much more complex situation. First, we studied the existence of pulse solutions and found regions in parameter space (including external tension, pulse velocity, and elastic parameters) where pulses exist (see Figures 1–3). Second we studied the stability of these solutions for extensible rod by taking advantage of the Hamiltonian nature of the amplitude equations. An analysis of the second variations of the Hamiltonian gives a complete description of the linear stability of these pulses as shown in Figures 4 and 5.

Appendix A

We show here that $H_{v,\omega}$ has only one negative eigenvalue and that the continuous spectrum consists of all positive real numbers. Let $\mathbf{v}_0 = (r_1, r_2, u_0, w_0, p_1, p_2, b_0, f_0)$ be the six-dimensional solution of (73) defined by a_0, b_0, and c_0 of (43) with

$$a_0 = p_1 + i p_2, \quad r_1 = -v\, p_1' - \omega\, p_2, \quad r_2 = -v\, p_2' + \omega\, p_1, \quad f_0 = \int (c_0 - 2\,|a_0|^2)\, d\xi .$$

Then, the first assumption made in [28] is that the linear operator

$$H_{v,\omega} \equiv \delta^2 I(\mathbf{v}_0) = \delta^2 E(\mathbf{v}_0) - v\delta^2 Q_r(\mathbf{v}_0) - \omega\delta^2 Q_g(\mathbf{v}_0) = \begin{pmatrix} D^2 & C \\ C^{\mathrm{T}} & L \end{pmatrix}, \tag{78}$$

where

$$D^2 = \begin{pmatrix} \frac{2P^2}{1+(1+\kappa)P^2} & 0 & 0 & 0 \\ 0 & \frac{2P^2}{1+(1+\kappa)P^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2\kappa(1+P^2)}{1+2P^2} \end{pmatrix}, \quad C = \begin{pmatrix} 2v\partial_s & 2\omega & 0 & 0 \\ -2\omega & 2v\partial_s & 0 & 0 \\ 0 & 0 & v\partial_s & 0 \\ 0 & 0 & 0 & v\partial_s \end{pmatrix} \tag{79}$$

and

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{12} & L_{22} & L_{23} & L_{24} \\ L_{13}^* & L_{23}^* & -\dfrac{\Gamma}{2}\,\partial_{ss} & 0 \\ L_{14}^* & L_{24}^* & 0 & -\kappa\,\frac{1+2\,P^2}{2\,(1+P^2)}\,\partial_{ss} \end{pmatrix}, \tag{80}$$

and

$$\begin{aligned} L_{11} = {} & 2(-\partial_{ss} - P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,b_0' + \kappa\,(1+2\,P^2)\,f_0' \\ & - 2\,P\,\Gamma\,\Psi\,\sqrt{1+\kappa\,P^2} + \big(P^2(\Gamma+2\,\kappa+\kappa\,P^2(1+\Gamma)) + \kappa\big)\big(p_2^2 + 3\,p_1^2\big)\big) \end{aligned} \tag{81a}$$

$$\begin{aligned} L_{22} = {} & 2(-\partial_{ss} - P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,b_0' + \kappa\,(1+2\,P^2)\,f_0' \\ & - 2\,P\,\Gamma\,\Psi\,\sqrt{1+\kappa\,P^2} + \big(P^2(\Gamma+2\,\kappa+\kappa\,P^2(1+\Gamma)) + \kappa\big)\big(p_1^2 + 3\,p_2^2\big)\big) \end{aligned} \tag{81b}$$

$$L_{12} = 4\,(P^2(\Gamma+2\,\kappa+\kappa\,P^2(1+\Gamma)) + \kappa)\,p_1\,p_2 \tag{81c}$$

$$L_{13} = 2\,P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,p_2\partial_s \tag{81d}$$

$$L_{14} = \kappa\,(1+2\,P^2)\,p_1\,\partial_s \tag{81e}$$

$$L_{23} = 2\,P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,p_2\partial_s \tag{81f}$$

$$L_{24} = \kappa\,(1+2\,P^2)\,p_2\,\partial_s \tag{81g}$$

and L_{ij}^* denoting the adjoint only has a finite number of negative eigenvalues. We will actually show there is only one negative eigenvalue. Writing an eight-dimensional function valued vector as $\mathbf{w} = (w_1, w_2)^{\mathrm{T}}$ where w_i, $i = 1, 2$ are two four-dimensional ones

$$\begin{aligned} \langle H_{v,\omega}\mathbf{w}, \mathbf{w}\rangle &= \langle D^2\,w_1, w_1\rangle + \langle C\,w_2, w_1\rangle + \langle C^T\,w_1, w_2\rangle + \langle L\,w_1, w_1\rangle \\ &= \langle D\,w_1, D\,w_1\rangle + 2\langle C\,w_2, w_1\rangle + \langle L\,w_1, w_1\rangle \\ &= \|D\,w_1 + D^{-1}C\,w_2\|^2 + \langle (L + C^2D^{-2})\,w_2, w_2\rangle, \end{aligned} \tag{82}$$

since $C^{\mathrm{T}} = -C$ and $CD = DC$. $\langle .,. \rangle$ is the L^2 inner product. If we define $L_1 = L + C^2D^{-2}$, then, by (82) and the minimax principle [30, 31], the problem is reduced to show that L_1 only has one negative eigenvalue. The linear operator L_1 takes the form

$$L_1 = \begin{pmatrix} L_2 & K \\ K^* & L_3 \end{pmatrix}, \tag{83}$$

where L_2 is a two-dimensional matrix of second-order linear differential operators and

$$K = \begin{pmatrix} 2\,P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,p_2 & \kappa\,(1+2\,P^2)\,p_1 \\ 2\,P\,\Gamma\,\sqrt{1+\kappa\,P^2}\,p_2 & \kappa\,(1+2\,P^2)\,p_2 \end{pmatrix}\partial_s \equiv \tilde{K}\,\partial_s, \tag{84a}$$

$$L_3 = \begin{pmatrix} \left(v^2 - \frac{\Gamma}{2}\right) & 0 \\ 0 & (v^2-1)\,\frac{\kappa\,(1+2\,P^2)}{2\,(1+P^2)} \end{pmatrix}\partial_{ss} \equiv M^2\,\partial_{ss}. \tag{84b}$$

We also reduce the problem on L_2 to a lower dimensional operator. Denote $\mathbf{y} = (y_1, y_2)^{\mathrm{T}}$, where y_i are two-dimensional vectors, then

$$\begin{aligned}\langle L_1\,\mathbf{y},\mathbf{y}\rangle &= \langle L_2\,y_1,\,y_1\rangle + \langle L_3\,y_2,\,y_2\rangle + 2\,\langle K\,y_1,\,y_2\rangle \\ &= \langle (L_2 - \tilde{K}^T\,\tilde{K})\,y_1,\,y_1\rangle + ||M\,\partial_s\,y_2 + \tilde{K}\,y_1||^2. \end{aligned} \tag{85}$$

Then we now have to prove that the operator

$$\mathcal{L} \equiv L_2\; - \tilde{K}^T\tilde{K} \tag{86}$$

only has one negative eigenvalue. This can be done by first diagonalizing $\mathcal{L}$. We introduce the change of variable

$$z_1 = \begin{pmatrix} \cos\left(\frac{\omega\,v\,(1+2P^2)\,s}{v^2+P^2(2v^2-1)}\right) & \sin\left(\frac{\omega\,v\,(1+2P^2)\,s}{v^2+P^2(2v^2-1)}\right) \\ -\sin\left(\frac{\omega\,v\,(1+2P^2)\,s}{v^2+P^2(2v^2-1)}\right) & \cos\left(\frac{\omega\,v\,(1+2P^2)\,s}{v^2+P^2(2v^2-1)}\right) \end{pmatrix} y_1 \tag{87}$$

Then

$$\langle \mathcal{L}\,y_1,\,y_1\rangle = \langle \tilde{\mathcal{L}}\,z_1,\,z_1\rangle \tag{88}$$

where

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{\mathcal{L}}_1 & 0 \\ 0 & \tilde{\mathcal{L}}_2 \end{pmatrix} \tag{89}$$

with

$$\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_2 + \Lambda, \tag{90a}$$

$$\begin{aligned}\tilde{\mathcal{L}}_2 = 2\,\Bigg[&\left(\frac{v^2 + P^2\,(2\,v^2-1)}{P^2}\right)\partial_{ss} - \frac{(1+2\,P^2)\omega^2(-v^2+4\,v^4P^2-6\,v^2P^2+P^2)}{(v^2+P^2)^2} \\ &- P\,\Gamma\,\sqrt{1+P^2}(2\,\Psi + b_0') + \kappa\,(1+2\,P^2)\,f_0' \\ &+\;2\,(P^2\,(\Gamma+2\,\kappa+P^2\,(1+\Gamma))+\kappa+1)\left(p_1^2+p_2^2\right)\Bigg], \end{aligned} \tag{90b}$$

where

$$\Lambda = -4\,\frac{a^2(1+P^2)(2\,v^4P^2 + 2\,P^2\Gamma\,v^4 + 2\,v^2P^2 - 3\,P^2\Gamma\,v^2 - P^2\Gamma - \Gamma\,v^2 + 2\,v^4)}{(-2\,v^2+\Gamma)(v^2-1)}$$

for $\kappa = 1$ and

$$\Lambda = -4\,\frac{P^2a^2(2\,\Gamma\,v^2 + 2\,v^2P^2 - P^2\Gamma + 2\,P^2\Gamma\,v^2)}{-2\,v^2+\Gamma}$$

for $\kappa = 0$.

The operator $\tilde{\mathcal{L}}_2$ has 0 as an eigenvalue since, from (38), (39), and (40), it is possible to show that

$$\tilde{\mathcal{L}}_2\,|a_0| = 0, \tag{91}$$

where a_0 is given in (43). Since the pulse has no zero, $\tilde{\mathcal{L}}_2$ does not have any negative eigenvalues. By taking the derivative of (90b) with respect to s we obtain

$$\tilde{\mathcal{L}}_1\,|a_0|' = 0. \tag{92}$$

Since $|a_0|'$ has one zero, $\tilde{\mathcal{L}}_1$ has one negative eigenvalue. Thus we have just demonstrated that $\tilde{\mathcal{L}}$ and hence $H_{v,\omega}$ only has one negative eigenvalue. The continuous spectrum of $H_{v,\omega}$ can be calculated using a well-known procedure ([32], Theorem A.2, p. 140). It is a straightforward calculation to show that the continuous spectrum of $H_{v,\omega}$ consists of the positive real numbers including 0.

Appendix B

The linear stability condition (77) in the case $\kappa = \chi = 1$ with $\omega = 0$ is equivalent to

$$\begin{aligned}
&\{(2\,v^4 - 3\,v^2\Gamma - \Gamma + 2\,v^4\Gamma + 2\,v^2)P^2 + 2\,v^4 - v^2\Gamma\}\\
&\times \{(-816\,v^{10}\Gamma^2 - 48\,v^6 - 120\,v^{10}\Gamma + 10\,\Gamma^3 - 12\,v^2\Gamma^2 + 160\,v^{12}\Gamma + 1140\,v^8\Gamma^2 - 860\,v^6\Gamma^2\\
&+ 228\,v^4\Gamma^2 + 336\,v^8 + 144\,v^{12} + 48\,v^{12}\Gamma^2 - 838\,v^6\Gamma^3 + 552\,v^8\Gamma^3 - 624\,v^{10} - 440\,v^6\Gamma\\
&+ 744\,v^8\Gamma - 150\,\Gamma^3v^2 + 72\,v^4\Gamma + 570\,\Gamma^3v^4 + 32\,v^{12}\Gamma^3 - 120\,v^{10}\Gamma^3)P^8\\
&+ (1428\,v^8\Gamma - 18\,v^2\Gamma^2 - 300\,v^{10}\Gamma - 508\,v^6\Gamma - 1250\,v^6\Gamma^2 + 3\,\Gamma^3 - 24\,v^6 + 360\,v^8 + 456\,v^8\Gamma^3\\
&- 36\,v^{10}\Gamma^3 + 294\,v^4\Gamma^2 + 1518\,v^8\Gamma^2 + 72\,v^{12}\Gamma^2 + 240\,v^{12}\Gamma + 36\,v^4\Gamma - 1224\,v^{10}\Gamma^2 - 1080\,v^{10}\\
&- 81\,\Gamma^3v^2 + 32\,v^{12}\Gamma^3 + 435\,\Gamma^3v^4 - 681\,v^6\Gamma^3 + 360\,v^{12})P^6\\
&+ (72\,v^4\Gamma^2 + 126\,\Gamma^3v^4 + 40\,v^{12}\Gamma^2 - 188\,v^6\Gamma^3 + 648\,v^8\Gamma^2 - 288\,v^{10}\Gamma + 888\,v^8\Gamma - 600\,v^{10}\Gamma^2\\
&+ 96\,v^8 + 112\,v^{12}\Gamma - 12\,\Gamma^3v^2 + 36\,v^{10}\Gamma^3 + 120\,v^8\Gamma^3 + 8\,v^{12}\Gamma^3 - 624\,v^{10} - 572\,v^6\Gamma^2 - 144\,v^6\Gamma\\
&+ 320\,v^{12})P^4 + (8\,v^{12}\Gamma^2 + 12\,v^8\Gamma^3 + 180\,v^8\Gamma + 16\,v^{12}\Gamma - 132\,v^{10}\Gamma - 23\,v^6\Gamma^3 - 90\,v^6\Gamma^2\\
&+ 114\,v^8\Gamma^2 + 15\,\Gamma^3v^4 - 120\,v^{10} + 120\,v^{12} + 12\,v^{10}\Gamma^3 - 96\,v^{10}\Gamma^2)P^2 + 12\,v^8\Gamma^2 + 16\,v^{12}\\
&- 24\,v^{10}\Gamma - 2\,v^6\Gamma^3\} > 0.
\end{aligned} \tag{93}$$

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