Toy models: The jumping pendulum

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We consider a simple pendulum consisting of a mass attached to an inextensible string of negligible mass. For small or large initial velocities, the motion of the pendulum is along a circle. When given sufficient but not too large an initial velocity, the mass will reach a certain height and leave the circle. After such a jump, it will follow a parabolic path until the string is again fully extended and the motion is again constrained by the string. We assume that the radial component (along the string) of the velocity of the mass instantaneously vanishes when the string becomes taut and that the mass loses some of its energy in the shock and resumes its circular motion. What is the dynamics of such a pendulum? Can it jump more than once? How many times can it jump?

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I. INTRODUCTION

We consider the motion in the vertical plane of a mass m labeled by P subject to a gravity force mg, linked to the origin O by an inextensible string of negligible mass and of length L (see Fig. 1). We take O to be the origin of a $\tilde{x}$-$\tilde{z}$ Cartesian plane, and define $-\mathbf{k}$ to be the direction of gravity ($\mathbf{i},\mathbf{j},\mathbf{k}$ are the unit vectors along the $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ axes).

We consider the case for which the initial conditions are such that the motion of the mass can be decomposed into circular and parabolic types of motion. In circular motion the string is extended and the mass follows the classical motion of a simple pendulum until it jumps.[1] The motion is parabolic from take-off until the string again becomes fully extended and the mass moves on a circular trajectory. The dynamics of this system is given by the superposition of these two types of motion. An equivalent mechanical system is the sliding of a mass inside a vertical circle. When the mass slides in the circle, it can jump, follow a parabolic trajectory, and land back on the circle.

II. CIRCULAR MOTION AND JUMPS

The differential equation for the motion of the mass is

$$m\tilde{\mathbf{a}} = -mg\mathbf{k} + \tilde{\mathbf{T}}, \tag{1}$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{v}}$ is the acceleration and velocity of the mass and $\tilde{\mathbf{T}}$ is the string tension. Before we proceed with our analysis, we simplify the formulation of the problem by introducing dimensionless variables in the following way:

$$t = \tilde{t}\sqrt{\frac{g}{L}}, \tag{2}$$

$$x = \frac{\tilde{x}}{L}, \quad z = \frac{\tilde{z}}{L}, \tag{3}$$

$$\mathbf{v} = \tilde{\mathbf{v}}\sqrt{\frac{1}{gL}}, \tag{4}$$

$$\mathbf{a} = \tilde{\mathbf{a}}\frac{1}{g}, \quad \mathbf{T} = \tilde{\mathbf{T}}\frac{1}{mg}. \tag{5}$$

In the new variables, the quantities t and x are dimensionless but play the same role as position and time. Similarly, $\mathbf{v}$ and $\mathbf{a}$ are the dimensionless acceleration and velocity; $\mathbf{T}$ is the dimensionless tension which is assumed to be in the normal direction, $\mathbf{T}\cdot\mathbf{v}=0$, and ensures that the distance of the mass to the origin is less than or equal to L in the $\tilde{x}$-$\tilde{z}$ plane (or, equivalently, the distance of the mass to the origin in the new x-z plane is less than or equal to 1). The magnitude of the vector $\mathbf{v}$ is denoted by v. In the new variables, Eq. (1) takes the simpler form

$$\mathbf{a} = -\mathbf{k} + \mathbf{T}. \tag{6}$$

In the rest of the paper, we use the dimensionless variables. For any given pendulum of mass m and string length L under a gravity force mg, the actual position, velocity, and acceleration can be found using Eqs. (2)–(5).

In the absence of friction the total dimensionless energy E,

$$E = \tfrac{1}{2}v^2 + z = \tfrac{1}{2}v_0^2 + z_0, \tag{7}$$

is conserved. That is, $v^2=2(h-z)$, where $h=E$ is a length determined by the initial position and velocity. If $h>1$, the velocity of the mass never vanishes, whereas for $h\leqslant 1$, the velocity of the mass vanishes when it reaches the height h. Because the constraint is given by a string and hence is unilateral, we need to know if the tension $\mathbf{T}$ vanishes at some point during the motion. If it does, the mass is subject only to gravity and follows a parabola.

To calculate $\mathbf{T}=T\mathbf{1}_n$, we project Eq. (6) onto the unit vector $\mathbf{1}_n$ along the outward normal to the circle, that is,

$$\mathbf{a}\cdot\mathbf{1}_n = -\mathbf{1}_z\cdot\mathbf{1}_n + T. \tag{8}$$

Because $\mathbf{a}=\dot{v}\mathbf{1}_t+v^2\mathbf{1}_n$ and $\mathbf{1}_n\cdot\mathbf{1}_z=-\cos\alpha=-z$, the tension is

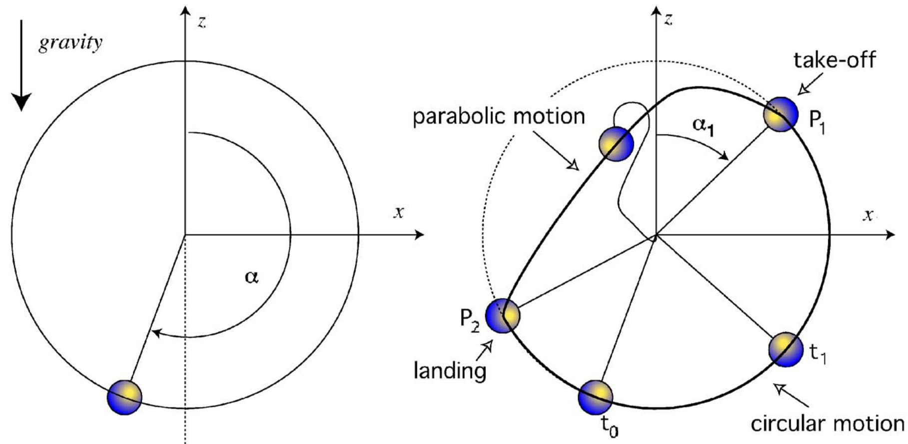

Fig. 1. The jumping pendulum is a regular pendulum with an inextensible string. We show a possible motion of the pendulum consisting of circular motion followed by parabolic motion until landing.

$$T = v^2 - z = 2h - 3z. \tag{9}$$

Equation (9) for the tension vanishes when $z=\frac{2}{3}h$. We conclude that if the initial conditions are such that h is either $-1 \leqslant h \leqslant 0$ (oscillatory motion) or $h \geqslant \frac{3}{2}$ (rotation), then the tension does not vanish during the motion, and the pendulum mass remains on a circular path indefinitely. However, if $0 < h < \frac{3}{2}$, the tension vanishes during the motion and the weight leaves the circle. The motion is then parabolic until the distance of the mass to the origin is again equal to 1. The string is then extended and the mass again follows a circular path. Note that our analysis is in terms of the energy and allows us to easily identify the take-off and landing points. The actual position of the trajectory of the pendulum as a function of time for large angles can be expressed in terms of elliptic functions (see Sec. V).

III. LANDING

The problem is to find the landing position of the mass and how it resumes its circular motion. To do so, we must make an assumption about the way the landing takes place. This assumption is a delicate point that requires careful examination. In this system the shock at landing is created through the sudden extension of the string. The shock will propagate in the string as a compressive wave that will eventually die out due to damping. A detailed analysis of the wave propagation of a slightly extensible string would be required to obtain a complete physical picture of the shock mechanisms (akin to problems arising in the sudden break of strings). However, the typical time scales of elastic wave propagation (of the order of the propagation of sound in air[2]) are very small compared to the time scales associated with the slow periodic motion of the pendulum, and we will therefore ignore these subtle effects to obtain a qualitative picture of the pendulum motion. Because at the moment of impact, the velocity may have a component normal to the circle, there is a collision, and we assume that at this time, the mass will keep only its tangential velocity and instantaneously lose its normal component which is absorbed by the sudden change of the string tension. Different reasonable assumptions for the new energy of the pendulum could be proposed and deserve further analysis. The model proposed here (the loss of the normal velocity) is based on the simplest physically relevant assumption (see Sec. VII for other interesting possibilities).

Let $(x_1, z_1) = (\sin\alpha_1, \cos\alpha_1)$ be the coordinates and angle of P_1, the position of the mass when it leaves the circle. We have $z_1 = \frac{2}{3}h$ with $0 < h < \frac{3}{2}$. From energy conservation, we have

$$v_1^2 = 2(h - z_1) = \tfrac{2}{3}h = z_1, \tag{10}$$

and $\mathbf{a}_1 = (-v_1\cos\alpha_1, v_1\sin\alpha_1)$. Therefore, the parabolic path followed by the mass after it leaves the circle can be parametrized by

$$x = x_1 - v_1 z_1 t, \tag{11a}$$

$$z = z_1 + v_1 x_1 t - \frac{t^2}{2}. \tag{11b}$$

The parabola intersects the circle when $x^2+y^2=1$, that is, $(x_1 - v_1 z_1 t)^2 + (z_1 + v_1 x_1 t - t^2/2)^2 = 1$, which after simplification reads

$$t^3\left(\frac{t}{4} - v_1 x_1\right) = 0. \tag{12}$$

Equation (12) shows that the unit circle is an osculating circle to the parabola (three points of contact) and intersects the parabola at time $t_2 = 4v_1x_1$, that is, at P_2 given by

$$(x_2, z_2) = (x_1 - 4x_1 v_1^2 z_1, z_1 + 4v_1^2 x_1^2 - 8v_1^2 x_1^2) \tag{13a}$$

$$= (\sin\alpha_1(1 - 4\cos^2\alpha_1), \cos\alpha_1(1 - 4\sin^2\alpha_1)), \tag{13b}$$

with the velocity (just before impact)

$$(v_x(t_2), v_z(t_2)) = (-v_1\cos\alpha_1, -3v_1\sin\alpha_1). \tag{14}$$

At the point of impact P_2, the unit tangent vector is $\mathbf{1}_t = -z_2\mathbf{1}_x + x_2\mathbf{1}_z$. The velocity just after impact is in the tangential direction $v_2\mathbf{1}_t$ and is given by the projection of the total velocity $\mathbf{v} = (v_x(t_2), v_z(t_2))$ before impact with the tangent vector $\mathbf{1}_t$, that is, using Eqs. (13) and (14),

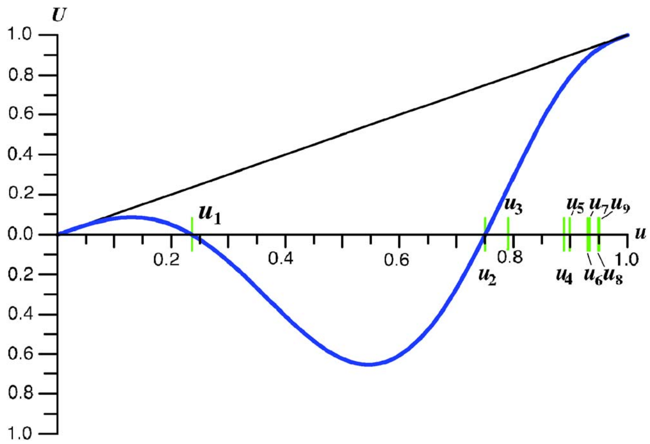

Fig. 2. The map giving U for a given u.

$$v_2 = v_1(1 + 4\sin^2\alpha_1 - 8\sin^4\alpha_1). \tag{15}$$

The energy E_2 of the mass at P_2 depends on v_2 and z_2, and is

$$E_2 = h_2 = z_2 + \frac{v_2^2}{2} = \cos\alpha_1\left(\tfrac{3}{2} - 32\sin^6\alpha_1\cos^2\alpha_1\right). \tag{16}$$

We note that the horizontal at P_1 bisects the angle between P_1P_2 and the tangent to the circle at P_1.

IV. REPEATED JUMPS

We now study if the energy E_2 is sufficient for the mass to leave the circle again. If we set $\cos\alpha_1 = u$ and $h_2 = \frac{3}{2}U$, we have $h_1 = \frac{3}{2}u$ and the relation between U and u is

$$U = u\left[1 - \tfrac{64}{3}(1-u^2)^3u^2\right]. \tag{17}$$

The energy before landing is $2u/3$ and the energy after landing is $2U/3$. Therefore, the map (17) gives the evolution of the energy as well as the height at which the pendulum jumps. The graph of $U(u)$ for $0 \leq u \leq 1$ is shown in Fig. 2; U is tangent to the bisector $U=u$ at $u=0$ and $u=1$. In this interval $U(u)$ has three zeroes located in increasing order at $u_0=0$, $u=u_1 \approx 0.24$, and $u=u_2 \approx 0.75$. It is a map that relates the height constant h_2 to h_1. The condition for a second take-off in the pendulum motion which follows the first collision is $h_2 > 0$, and hence $U > 0$. Because $h_2 < h_1$, the condition $h_2 < \frac{3}{2}$ is obviously satisfied. Thus, the condition for a second jump is that $0 < u < u_1$ or $u_2 < u < 1$.

The possibility of a third jump can then be investigated using the same graph. Starting, for instance, with a value of u in the interval $(u_2, 1)$, the corresponding value of U is immediately visualized on the graph (see Fig. 2), and is used as a new value of u which can easily be positioned on the u axis using the bisector $U=u$. This new value of u determines whether or not a third jump occurs. If it does, the same procedure is then to be repeated (Fig. 3).

To classify all the possibilities for the number of succes-

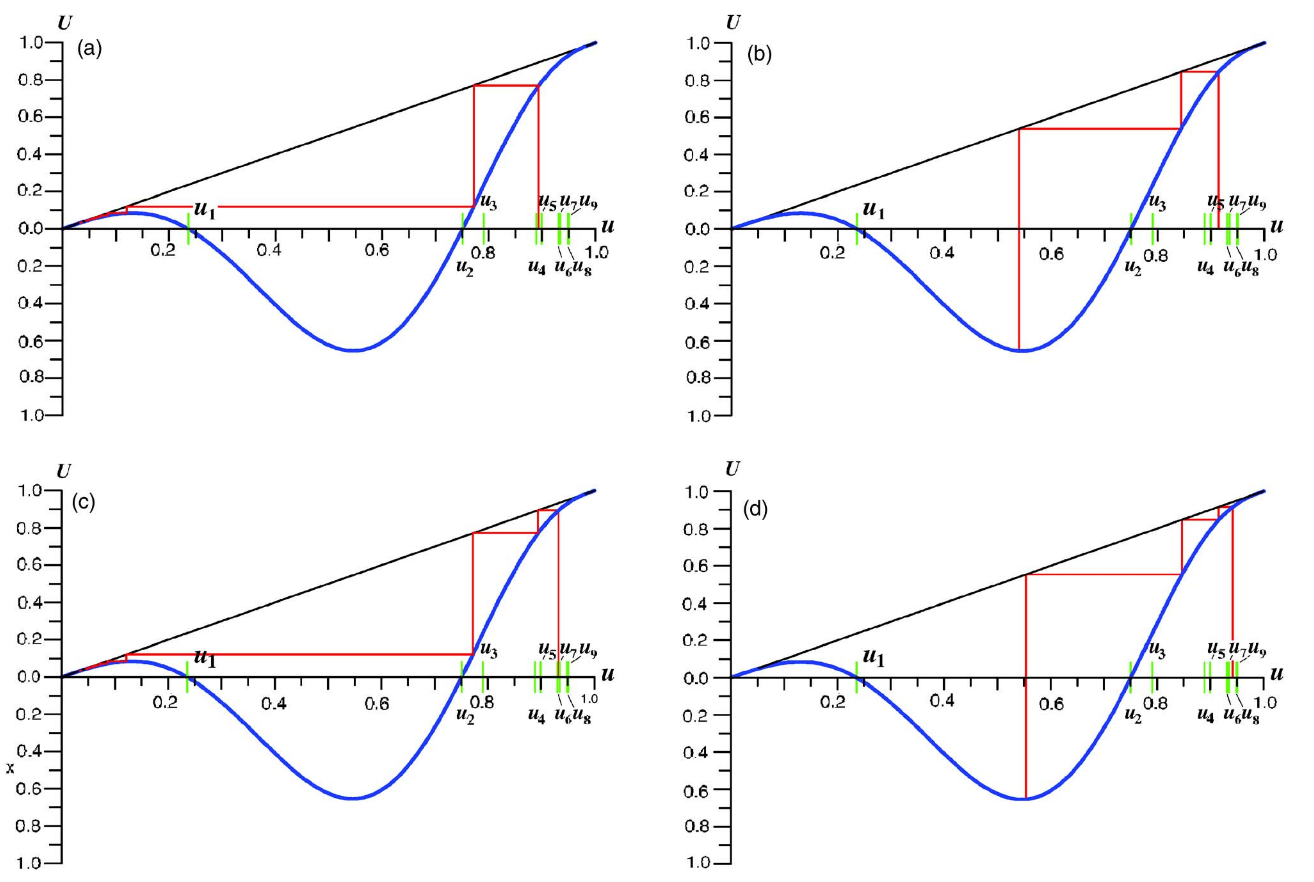

Fig. 3. Different possible scenarios of jumps and landing. The initial value for u is chosen to be (a) $(u_4+u_5)/2$, (b) $(u_5+u_6)/2$, (c) $(u_6+u_7)/2$, and (d) $(u_7+u_8)/2$.

sive jumps, we define for $n>2$, the iterates of u as $U_n = U^n(u)$ and u_n as the preimage of u_{n-2}, that is,

$$u_{n-2} = U(u_n), \tag{18}$$

or equivalently

$$u_n = U^{-1}(u_{n-2}). \tag{19}$$

Thus, starting with u_1 and u_2, Eq. (19) defines a monotonically increasing sequence u_n, with $\lim_{n\to\infty} u_n = 1$ (see Fig. 2). In practice, u_1 and u_2 are first obtained as the two roots of $U(u)$ in the interval $(0,1)$ and the other values of u_n are obtained by root solving iteratively $u_{n-2}=U(u_n)$ in the interval $(0,1)$. The first few values are $u_1 \approx 0.2356$, $u_2 \approx 0.7506$, $u_3 \approx 0.7908$, $u_4 \approx 0.8889$, $u_5 \approx 0.8993$, $u_6 \approx 0.9305$, and $u_7 \approx 0.9346$.

We classify all the possible motions.

1. The condition $0<u<u_1$, $0<U_1<u<u_1$ implies $0<U_n<u<u_1$ and the mass jumps infinitely many times. The point $u=0$ is a stable fixed point which is attracting for all $u \in [0,u_1]$. The motion takes place close to the intersection of the circle with the x axis and consists of a little jump followed by an oscillation to the other side of the circle not unlike a skateboarder in a half-tube.
2. For $u_1 \leqslant u \leqslant u_2$, $U_1<0$ and the mass jumps only once and then oscillates on the lower circle.
3. If $u_2<u<u_3$, noting that $U(u)$ is monotonically increasing for $u_2<u<1$, we have $U(u_2)<U_1<U(U^{-1}(u_1))$, that is, $0<U_1<u_1$. After the first jump, the mass has an energy in the first interval and jumps indefinitely (see case 1).
4. If $u_3 \leqslant u \leqslant u_4$, then $U^{-1}(u_1) \leqslant u \leqslant U^{-1}(u_2)$, that is $u_1<U_1<u_2$. The mass jumps twice and then oscillates.
5. If $u_{2n}<u<u_{2n+1}$, then after n jumps, the mass has an energy such that U_n lies in (u_0,u_1) and then jumps indefinitely as in the first case [see Fig. 4(a)].
6. If $u_{2n+1} \leqslant u \leqslant u_{2n+2}$, the mass jumps $(n+1)$ times before oscillating on the circle [see Fig. 4(b)].

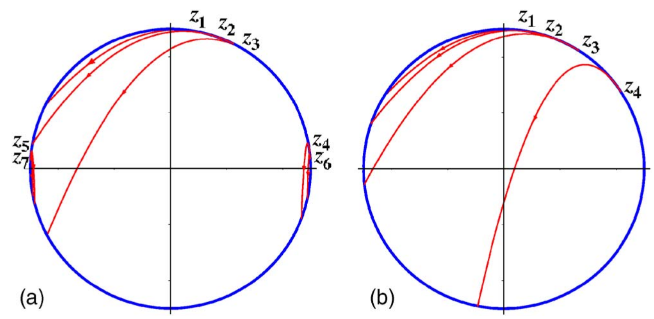

Fig. 4. Different possible scenarios of jumps and landings. The initial value for u is chosen to be (a) $(u_6+u_7)/2$ and (b) $(u_7+u_8)/2$, corresponding to the two lower plots in Fig. 3.

V. TIME BETWEEN JUMPS

We now calculate the time between two jumps specified by u and U. The time is again decomposed into the flight time t_f and the time during the pendular motion t_p. According to Eq. (12), the flight time is given by

$$t_f = 4 v_1 x_1 = 4\sqrt{u(1-u^2)}. \tag{20}$$

The pendulum time is more delicate to calculate. We need to find the time taken by a pendulum with energy $h_2 = \frac{3}{2}U$ to go from the angle at impact θ and the angle at take-off α_2. The angle at impact is determined by the height $z_2 = \cos\theta$ given by Eq. (13), that is,

$$\cos\theta = \cos\alpha_1(1 - 4\sin^2\alpha_1) = u(4u^2-3). \tag{21}$$

The time t_p is obtained by considering energy conservation, $E=\frac{1}{2}v^2+z$, written in terms of the angle $\alpha=\alpha(t)$, that is,

$$\frac{1}{2}\left(\frac{d\alpha}{dt}\right)^2 + \cos\alpha = E, \tag{22}$$

which after inversion and integration becomes

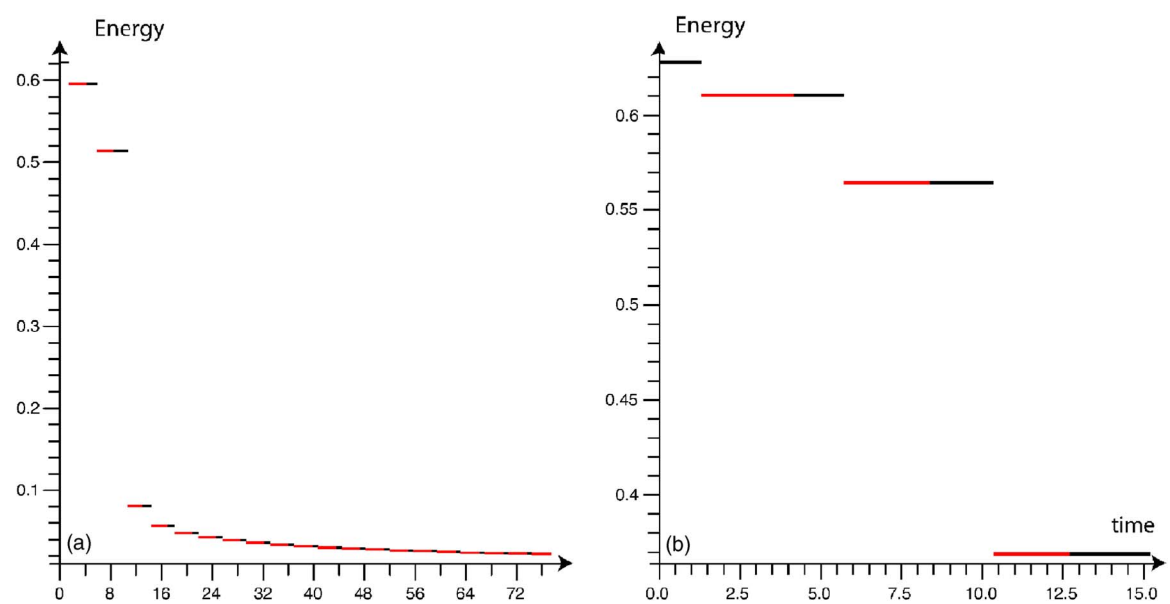

Fig. 5. The energy as a function of time. At each jump, a fraction of the energy is lost. The flight time and the time between jumps are calculated according to Eqs. (20) and (25). The two graphs correspond to the initial conditions in Fig. 4, that is, (a) $(u_6+u_7)/2$ and (b) $(u_7+u_8)/2$.

$$t = \epsilon \int \frac{d\alpha}{\sqrt{2(E - \cos \alpha)}}, \tag{23}$$

where $\epsilon^2 = 1$ is chosen for the time to be positive. For our particular problem, we have

$$t_p = \epsilon \int_{\theta}^{\alpha_2} \frac{d\alpha}{\sqrt{3U - 2 \cos \alpha}}, \tag{24}$$

and $\epsilon = -1$ if $\alpha_2 < \theta$ and $\epsilon = +1$ otherwise. The integral can be integrated explicitly in terms of the incomplete elliptic integral of the first kind

$$t_p = k \left| F\left(\cos \frac{\theta}{2}, k\right) - F\left(\cos \frac{\theta_2}{2}, k\right) \right|, \tag{25}$$

where $k = 2/\sqrt{3U+2}$ and θ and α_2 are given explicitly in terms of u and U. We thus have an explicit formula for the time between the jump at height u and at height U. As an example, we have computed the flight and pendular times for the initial condition given in Fig. 4 and plotted the evolution of the energy in Fig. 5.

VI. CONCLUSIONS

We now have a complete picture of the dynamics of the jumping pendulum. If the initial energy of the mass is such that $u_{2n} < u < u_{2n+1}$, the pendulum will jump infinitely many times (n large jumps crossing the z axis followed by infinitely many small jumps crossing the x axis). If the initial energy is such that $u_{2n} \leq u \leq u_{2n+1}$, the pendulum will jump $n+1$ times (large jumps) and then oscillate.

VII. SUGGESTED PROBLEMS

1. In light of the discussion on jumps, is it reasonable to believe (as some people remember from their childhood) that it is possible to perform a complete rotation on a classic swing? Why are circus swings attached with bars rather than ropes? See Ref. 3 for an interesting discussion on how to pump a swing.
2. If we constrain the mass to follow a closed curve other than a circle, is there a curve such that for particular values of the energy, the mass will jump and land without loss of energy? That is, is there a curve for which the ballistic parabola is tangent at two points (open problem)?
3. Another model of impact is to assume that the mass keeps a constant fraction α of its energy. Describe the motion.
4. The map (17) is restricted to the interval from $u=0$ to $u=1$. If $U(u) < 1$, the motion changes. Although physically irrelevant, it is of interest to study the dynamics of the map (17) as a first-return map from the interval $[-1,1]$ onto itself. Show that the system displays chaotic features by tracking the evolution of a set of trajectories. For more information on chaotic motion in first-return maps see Ref. 4.
5. Consider the map for the model in Problem 3. Study the motion as a function of α and identify the critical value of α at which the motion first becomes chaotic. How does it relate to our understanding of the logistic map?
6. Model a jumping forced and damped pendulum by assuming that the motion of the pendulum is damped but forced (for example, magnetically) in its pendular motion.[5] Is the motion chaotic?

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THE DISCOVERY OF NEW SPACE DIMENSIONS

If superstring theory is proven correct, we will be forced to accept that the reality we have known is but a delicate chiffon draped over a thick and richly textured cosmic fabric . . . determining the number of space dimensions—and, in particular, finding that there aren't just three—would provide far more than a scientifically interesting but ultimately inconsequential detail. The discovery of extra dimensions would show that the entirety of human experience had left us completely unaware of a basic and essential aspect of the universe. It would forcefully argue that even those features of the cosmos that we have thought to be readily accessible to human senses need not be.

Brian Greene, *The Fabric of the Cosmos: Space, Time, and the Texture of Reality* (Knopf, 2004), p. 19.