Painlevé Analysis and Integrability

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Abstract

In this report the method of Painlevé analysis is introduced and explored. Lax pairs for systems of differential equations are also introduced and their relationship to problems of integrability are shown. Then another technique of singularity analysis is introduced. A specialization of the Painlevé analysis that the report began with.

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1 Introduction

This paper is a short personal view of integrability and Lax pairs. There are some relationships between general properties of Lax pairs and Painlevé analysis. In section 2, I review general notions regarding Painlevé analysis. Section 3 is a quick overview of what constitutes a Lax Pair and why we should consider them. I also give Lax pairs of some simple linear systems as well as two more complicated ones from the literature. Little progress was made in the question as to how to find Lax pairs or their relationship with the Painlevé analysis other than that given by the (WTC) method [5]. Finally I also consider the method of converting analysis around singularities into a problem in dynamical systems, this method and some of its implications are presented in section 4.

The results presented in this paper are attempting to find a property of certain differential equations, called *integrability.* This word has many meanings for what we are doing, depending on which method we are using. One notion of integrability, which Painlevé analysis examines, is the existence of solutions as Laurent series on open sets of the complex time variable. Integrability demands that these solutions are consistent, or match on the overlapping pieces of the sets on which they are defined. In other words this notion of integrability is existence of meromorphic solution.

The other notion of integrability, where Lax pairs play a central role is related to the existence of a "full" set of independent constants. For instance, for a Hamiltonian system in $2n$ variables we demand that the system admits n independent constants of the motion. Such constants will be referred to as first integrals.

2 Introduction to Painlevé Analysis and Integrability

In the study of integrability, in the sense of series expansions of the solutions, some local properties of the system of equations have been extended to answer the global question of integrability. In this paper I will present the ideas behind one type of local analysis, and show a couple of different methods related to it. The study centers around an examination and classification of the types of singularities which are exhibited by the general and in some cases particular solutions of the system.

Definition 1 *A system of ordinary differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^m, \quad t \in \mathbb{R}$$

is said to have the Painlevé property if the general solution has no movable critical singularities.

Movable, in the above definition refers to the arbitrary position of the solution's singularities in complex time. For any solution the presence and position of movable singularities is given by the initial conditions. The other type of singularities that can be found are *fixed singularities*, and they are a consequence of a time singularity in the function $\mathbf{f}$. These singularities are fixed in time however so we may make the stipulation that we are examining the solutions outside of a neighborhood containing them, or introduce the proper branch cuts or a change of variables.

The Painlevé property also has a PDE version [5], our focus is on ODEs in this paper. The Painlevé property is related to the meromorphic properties of solutions to a system of ordinary differential equations.

In an attempt to decide if a system has the Painlevé property (A difficult question we can only reach a partial answer on) we will expand a formal Laurent series solution about an arbitrary singular point in complex time. We begin with a leading order analysis for the homogeneous system of equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}). \tag{1}$$

where the components of $\mathbf{f}$ are polynomial in the dependent variables x_i. We substitute into this equation the ansatz

$$x_i = a_i(t - t_*)^{p_i}, \qquad i = 1, 2, ..., n \tag{2}$$

Where t_* is the position of the singularity, for ease of use we set $\tau = (t - t_*)$. So that this is an honest singularity, we will require that at least one $p_i < 0$ but this requirement is always satisfied except for some linear systems. After the substitution we get the equation

$$p_i a_i \tau^{p_i - 1} = f_i(a_1\tau^{p_i}, ..., a_n\tau^{p_n}), \qquad i = 1, 2, ..., n \tag{3}$$

We solve this equation for the p_i's and a_i's which will balance the leading order terms (terms with a large negative exponent of τ). What we end up with if the solutions are found is a number of different leading orders, which we will now use to construct a possible general solution in a neighborhood of the singularity t_*.

For instance we consider the equation

$$\ddot{x} = 2x^3. \tag{4}$$

We begin with the ansatz $x = a\tau^p$,

$$p(p-1)a\tau^{p-2} = 2a^3\tau^{3p} \tag{5}$$

First we set the two powers of τ equal and solve for p to get $p = -1$. Next we solve for a to get $a = \pm 1$. We have two possible leading terms however both involve the same power of τ so in the Laurent expansion we will condense them to one equation.

Next, for the equation

$$\dot{x} = 3x^2. \tag{6}$$

We begin with the ansatz $x = a\tau^p$,

$$pa\tau^{p-1} = 3a^2\tau^{2p} \tag{7}$$

So first we set the two powers of τ equal and get $p = -1$, then we solve for a to get $a = -\frac{1}{3}$. So we get only one leading term and it is of integer order.

For the Laurent series expansion of the solution we want to guarantee convergence in a neighborhood of the singularity. To this end the analysis just performed has determined whether the exponent p_i is an integer that can be the first term in a Laurent series.

If the p_i's have non-integer values, these represent singularities that will require a branch cut to expand solutions near them, this sounds bad but in some cases the solution can still be continued around these types of singularities making them non critical. We will not address this case in this paper.

In order to build a Laurent series solution, we make the substitution of the formal series

$$x_i = \sum_{j=0}^{\infty} a_{ij}\tau^{j+p_i} \tag{8}$$

Setting coefficients of corresponding powers of τ equal we get a set of compatibility conditions for the series solution.

What can we hope for? At best, we have a consistent set of compatibility conditions with enough arbitrary coefficients so that the series describes the local expansion of the general solution. The powers of j whose corresponding coefficients a_{ij} are arbitrary are called *resonances* of the system.

If the series does not contain all the arbitrary coefficients or are not consistent, we are forced to include logarithm terms in the expression.

$$x_i = \sum_{j=0}^{\infty} a_{ij}\tau^{p_i}(\log\tau)^j, \qquad i = 1, ..., m \tag{9}$$

these series will require branch cuts to be well defined. Once again we must set aside these considerations until a later time.

What does all this have to do with the Painlevé property? A solution whose Laurent expansion converges in a neighborhood of the singularity is called *regular*. If a branch cut is necessary to define the expansion the singularity is *branched* or *critical*: branched if the expansion involves the necessary number of arbitrary constants; critical if the expansion does not converge, or the conditions on the coefficients are inconsistent.

Here we concentrate on finding regular singularities and exploring their solutions. If a singularity is regular then the general solution near the singularity can be written as the Laurent series. If all movable singularities are regular then the general solution is always extendable in the neighborhood of a singularity.

The Painlevé property helps determine integrability and guarantees it for solutions in a local neighborhood.

3 Lax Pairs

We will be defining matrices whose entries are functions of the dependent variables of a system for ordinary differential equations denoted $A = A(\mathbf{x})$. These functions depend on time so we can consider taking their time derivative.

Definition 2 *The time derivative of a matrix* $A = A(\mathbf{x})$ *along the flow defined by the vector field* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *is defined as:*

$$\left(\frac{dA}{dt}\right)_{ij} = \frac{dA_{ij}}{dt} = \dot{\mathbf{x}} \cdot \partial_x A_{ij} = \mathbf{f} \cdot \partial_x A_{ij}, \qquad i, j = 1, ..., m \tag{10}$$

Definition 3 *A Lax pair for a vector field* $\dot{\mathbf{x}} = f(\mathbf{x})$ *is an ordered non-constant pair of matrices* (A, B) *such that*

$$\frac{dA}{dt} = [B, A] \tag{11}$$

Where the $[,]$ represents the matrix commutator, $[B, A] = BA - AB$. We will see that this equation is equivalent to requiring A to fall in the same conjugacy class of matrices with function entries along the flow defined by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

For an example we will use the system

$$\begin{aligned} \dot{u}_1 &= u_1(u_2 - u_3) \\ \dot{u}_2 &= u_2(u_3 - u_1) \\ \dot{u}_3 &= u_3(u_1 - u_2). \end{aligned} \tag{12}$$

Equations of this type show up often when we work backwards from a simple matrix L and try to find the system it could be the Lax pair matrix for.

van Tonder [3] found the Lax pair

$$A = \begin{pmatrix} 0 & 1 & u_1 \\ u_2 & 0 & 1 \\ 1 & u_3 & 0 \end{pmatrix}, B = \begin{pmatrix} u_1 + u_2 & 0 & 1 \\ 1 & u_2 + u_3 & 0 \\ 0 & 1 & u_3 + u_1 \end{pmatrix} \tag{13}$$

The surprising fact about Lax pairs is that they have the possibility of giving a number of first integrals, which relates to one of the notions of integrability (see section 1).

Proposition 4 *The eigenvalues of* A *are constants of the motion.*

Indeed, let λ be the eigenvalue of an eigenvector ψ.

$$A\psi = \lambda\psi \tag{14}$$

The Lax pair relation implies

$$B\psi = \psi_t. \tag{15}$$

Differentiating (14), we get

$$\frac{dA}{dt}\psi + A\psi_t = \lambda_t\psi + \lambda\psi_t \tag{16}$$

which together with (11) leads to

$$BA\psi - AB\psi + A\psi_t = \lambda_t\psi + \lambda\psi_t \tag{17}$$

$$\lambda B\psi - A\psi_t + A\psi_t = \lambda_t\psi + \lambda\psi_t \tag{18}$$

Hence, we have

$$0 = \lambda_t\psi \tag{19}$$

Corollary 5 *The trace and determinant of A are also first integrals of the system*

Corollary 6 *If the matrix A is diagonalizable then we find that for all powers of A the eigenvalues and so the trace and determinants are first integrals of the system.*

For our example we get that the

$$\det(A) = u_1u_2u_3 \tag{20}$$

and

$$\det(A^2) = u_1 + u_2 + u_3. \tag{21}$$

So the Lax pair yields two first integrals of the system 12.

There is no known general algorithm for finding Lax pairs for arbitrary ODEs. There is a method (WTC) [5] for the equivalent problem in PDE's which finds equations corresponding to the compatibility conditions found in the Painlevé analysis, a choice of basis for the functions space, allows us to convert these equations to the matrix form given before.

We can extend the Lax pairs we just defined to pairs of matrices that will enable us to find an entire curve of constants by introducing a spectral parameter λ, to get the equation

$$\frac{dA_\lambda}{dt} = [A_\lambda, B_\lambda] \tag{22}$$

This has a number of advantages, the obvious one being that the arbitrary parameter λ may allow us to coax useful first integrals out of the Lax pair. For instance it can be shown that the coefficients of the terms in (23) are constants of the system. Now, consider the curve (λ, μ) defined by the relation

$$\det(A_\lambda - \mu I) = 0. \tag{23}$$

One of the most remarkable result of the theory of of Lax pairs is that information about the Liouville tori in Hamiltonian dynamics can be found from the topological and geometrical properties of this curve. (See for instance the book on this technique by Michèle Audin, where these methods are applied to spinning tops [6]).

We will now present some more examples of Lax pairs. Unfortunately these examples are “cooked” that is I found some of the first integrals of the system and put together a matrix that would preserve those. You will not believe how hard these matrices are to find until you try to find a couple yourself. There is a wide range of literature on finding Lax pairs but so far no satisfactory answer to the question. Some problems from mechanics exhibit interesting Lax pairs, and some systems even admit only a Lax pair without a spectral parameter, the significance of which is related to the relationship between the spectral curve and the Liouville tori.

For the system,

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$$

We have a Lax pair

$$A = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & \frac{1}{y} \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & cyx & dx \\ d & b-1 & c \\ cy & dxy & b \end{pmatrix} \tag{24}$$

For the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

We have a Lax pair

$$A = \begin{pmatrix} 0 & -\lambda & x \\ \lambda & 0 & -y \\ y & x & \frac{1}{\lambda} \end{pmatrix} \tag{25}$$

The B matrix is not very important, it only contains information on how the matrix A remains in the same conjugacy class. The important results involve properties of A. These two examples demonstrate the pattern for linear systems. I made it a little farther in generalizing to a linear system involving terms in x and y but the results are not complete.

Finally, there are some classical examples of Lax pairs. From Bobenko and Kuznetsov [2] we have a Lax pair for the Goryachev-Chaplygin top. The Goryachev-Chaplygin top is the system given by the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1 \tag{26}$$

with the variables of angular momentum $M = (M_1, M_2, M_3)$ and the field strength in the moving frame $p = (p_1, p_2, p_3)$.

We get the Lax pair

$$A = \begin{pmatrix} \frac{2}{3}i\gamma & -ip_3/\lambda & -M_2 + iM_1 \\ ip_3/\lambda & -2iM_3 - \frac{4}{3}i\gamma & -2i\lambda + (p_2 - ip_1)/\lambda \\ M_2 + iM_1 & (p_2 + ip_1)/\lambda + 2i\lambda & 2iM_3 + \frac{2}{3}i\gamma \end{pmatrix} \tag{27}$$

$$B = \begin{pmatrix} -3iM_3 - \frac{2}{3}i\gamma & 0 & -M_2 + iM_1 \\ 0 & -2iM_3 - \frac{2}{3}i\gamma & -2i\lambda \\ M_2 + iM_1 & 2i\lambda & 2iM_3 + \frac{4}{3}i\gamma \end{pmatrix} \tag{28}$$

Where λ is the spectral parameter. This Lax pair is used by Bobenko and Kuznetsov to expand the solution of the Goryachev-Chaplygin top using methods of algebraic geometry and Baker-Akhiezer functions. This is definitely something that should be examined further.

Lax pairs have been used to extend the equations of spinning tops into arbitrary dimensions. They provide generalizations of many other problems as well. An interesting result using Lax pairs associates to each integrable system in three dimensions a corresponding motion of a filament in three dimensions.

The Lax pair formulation is closely tied to the singularity analysis we began this paper with, as is demonstrated by the PDE versions of Lax pairs. So it does appear that the two should be related in some way. In conclusion for this section, there is still much that needs to be done.

4 Linearization of the Singular Manifold

We extend the leading order analysis the paper began with to an analysis of the resonant terms in the Laurent expansion. What we do in this section converts the singularity analysis that we have introduced into a linearization problem of a dynamical system. Equivalently the problem could be worked backwards, that is finding the singularity problem corresponding to a dynamical one.

The method is simple, for a homogeneous system of ODE's

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R} \tag{29}$$

with the components of $\mathbf{f}$ polynomial in the dependent variables.

We make the ansatz

$$x_i = \tau^{p_i}(a_i + \phi_i(\tau)). \tag{30}$$

With the series expansions we want to consider corresponding to $\phi(\tau) \to 0$ as $\tau \to 0$. The a_i and p_i appearing above are the resulting expressions from the leading order analysis. After substitution of (30) into (29) and further simplification, we obtain

$$\tau\dot{\phi} = (D\mathbf{f}(\mathbf{a}) - \operatorname{diag}(\mathbf{p}))\phi(\tau) + h(\phi) \tag{31}$$

where $h(\phi)$ is the upper tail of the Taylor series for $\mathbf{f}$. Making the substitution $\tau = e^s$, we obtain

$$\frac{d\phi}{ds} = (D\mathbf{f}(\mathbf{a}) - \operatorname{diag}(\mathbf{p}))\phi(\tau) + \mathbf{h}(\phi(\tau)) \tag{32}$$

So we get a differential equation in ϕ.

We wish to examine this system in a neighborhood of $\tau = 0$, and $\phi = 0$, corresponding to $s \to -\infty$. The Taylor expansion of ϕ is a parametrization of the unstable manifold of $\phi = 0$. In order to understand the relationship between the unstable manifold and the series solutions let us first consider the linearized system around $\phi = 0$.

We get solutions of the form $\phi(s) = e^{\lambda s}\phi_0$ for an eigenvalue $\lambda > 0$. This solution, converting back to the τ variable is $\phi(\tau) = \tau^{\lambda}$. What have we done? In the Laurent Expansion about τ we have set all coefficients except the arbitrary ones to be zero. So we are left with the terms of the resonances, it is precisely these terms that the linearization of ϕ has returned. The power of this method is that it picks out the non-integer resonances as well as the integer ones we found using the Laurent expansion. Therefore we can find a larger class of general solutions and extend the possibility of an expanded solution in a neighborhood of a singularity. It enables a more powerful method of testing for the proper number of resonances.

However, we must still watch for expansion terms which involve terms of τ with negative fractional exponents, these type of terms will require branch cuts.

From here we have many interesting questions to explore, what do the other possible solutions to the linearization problem correspond to when we map them back to the singularity problem. If we drop the requirement that $\phi(\tau) \to 0$ as $\tau \to 0$ then we are allowed to have singularity solutions that are from the stable manifolds of the dynamical system.

We will examine a solution of the linearization of (32) that is of the form $\phi(s) = e^{\lambda s}\phi_0$, where $\lambda < 0$. We get $\phi(\tau) = \tau^{\lambda}\phi_0$. So when we look back to the solution near the singularity, (30), we see that we have the solution

$$x_i = a_i\tau^{p_i} + \phi_0\tau^{\lambda+p_i} \tag{33}$$

So we have found a solution which has a Laurent expansion with leading order below, in the sense of degree ordering of the terms of the series the initial leading order we started with. It needs to be examined if these terms are other leading orders of the system or if they are an entirely new set of expansions.

We can also examine solutions corresponding to sources, sinks, and centers, to get solutions of the form

$$x_i = \tau^{p_i}a_i + \phi_{0i}\tau^{u+p_i+vI} \tag{34}$$

If the linearization contains repeated eigenvalues we may get solutions of the form,

$$\phi_i(s) = e^{\lambda s}q_i(s)\phi_{0i}. \tag{35}$$

For initial conditions ϕ_0, q_i and polynomial in s.

When we change back to τ coordinates we get a solution

$$\phi_i(\tau) = \tau^{\lambda}q_i(\log\tau)\phi_0 \tag{36}$$

So we have uncovered the logarithm terms that would be necessary in the series expansion. A repeated resonance may force the inclusion of logarithm terms in the expansion.

Finally we consider a system of the form

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned} \tag{37}$$

For polynomial f and g, such that the ansatz

$$\begin{aligned}x &= a(\log\tau)^p\tau^l\\ y &= b(\log\tau)^q\tau^r\end{aligned} \tag{38}$$

is consistent.

So we make the substitution

$$\begin{aligned}x &= (\log\tau)^p\tau^l(a+\phi(\tau))\\ y &= (\log\tau)^q\tau^r(b+\psi(\tau))\end{aligned} \tag{39}$$

After simplifying and rearranging we have the equations,

$$\begin{aligned}\tau(\log\tau)^p\dot{\phi} + p(\log\tau)^{p-1}\phi &= (Df(x,y) - (l(\log\tau)^p, 0))\begin{pmatrix}\phi\\ \psi\end{pmatrix} + h_1(\phi,\psi)\\ \tau(\log\tau)^q\dot{\psi} + q(\log\tau)^{q-1}\psi &= (Dg(x,y) - (0, r(\log\tau)^q))\begin{pmatrix}\phi\\ \psi\end{pmatrix} + h_2(\phi,\psi)\end{aligned} \tag{40}$$

We can now change variables via $\tau = e^s$

$$\begin{aligned}\frac{d(s^p\phi)}{ds} &= (Df(x,y)\mathrm{diag}(1/s^p, 1/s^q) - (l,0))\begin{pmatrix} s^p\phi \\ s^q\psi \end{pmatrix} + h_1(\phi,\psi) \\ \frac{d(s^q\psi)}{ds} &= (Dg(x,y)\mathrm{diag}(1/s^p, 1/s^q) - (0,r))\begin{pmatrix} s^p\phi \\ s^q\psi \end{pmatrix} + h_2(\phi,\psi) \qquad (41)\end{aligned}$$

So we are again left with a linearization problem, the change of variables to reach the original equation is a little more complicated this time.

We now return to considering the ansatz (30). If $\phi(\tau)$ is not analytic, we expect that the solutions of the linear system will not correspond to the actual equation near the singularity. The idea of this analysis is looking at the geometric and analytic "shapes" of solutions near a movable singularity.

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