EIGENVALUE PROBLEM AND A NEW PRODUCT IN THE COHOMOLOGY OF FLAG VARIETIES

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Let G be a complex semisimple algebraic group and let K be a maximal compact subgroup with their Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Choose a maximal subalgebra (which is necessarily abelian) $\mathfrak{a} \subset \mathfrak{p}$ and let $\mathfrak{a}_+$ be a dominant chamber in $\mathfrak{a}$. Then any $K-$orbit in $\mathfrak{p}$ intersects $\mathfrak{a}_+$ in a unique point.

For any $n \geq 2$, the celebrated *Hermitian eigenvalue problem* concerns determining the following subset Δ_n of $\mathfrak{a}_+^n$:

$$\Delta_n := \{(a_1, \dots, a_n) \in \mathfrak{a}_+^n : \exists (x_1, \dots, x_n) \in \mathfrak{p}^n \text{ with } \sum x_i = 0 \text{ and } x_i \in AdK.a_i\}.$$

By works of several mathematicians including Klyachko, Berenstein-Sjamaar, Belkale, Δ_n is given by certain inequalities parametrized by standard maximal parabolic subgroups P of G and n Schubert cohomology classes $\epsilon_{w_1}^P, \dots, \epsilon_{w_n}^P$ such that

$$\epsilon_{w_1}^P \cdot \cdots \cdot \epsilon_{w_n}^P = \epsilon^P,$$

where ϵ^P is the top cohomology class of G/P.

But, as shown by Kumar-Leeb-Millson, these set of inequalities are, in general, not irredundant.

Now, the main topic of this talk is a recent joint work with Belkale. We give a new commutative and associative product in the cohomology $H^*(G/P)$ of any flag variety G/P (which still satisfies the Poincaré duality) and show that the ineqalities determining Δ_n are given in terms of this new product in $H^*(G/P)$ for maximal parabolics P. This results in general in far fewer inequalities determining Δ_n. We show that for simple groups of rank 3, our new set of inequalities is an irredundant system.

We believe that similar results can be obtained for the cone which determines when the product of n elements in K is 1 in terms of the modified product in the quantum cohomology of G/P's.