Answers To Test 3

Math 250a, Section 2
20 November 2007
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NAME (please print) : ________________________________

Instructions. Do all problems. **Show your work and justify answers** in order to receive full credit. Use of textbook, calculators, and notes is allowed. Total points = 100.

1. (12 points)

(**a**) Does the sequence

$$a_n = \frac{2n+1}{n}, \qquad n = 1, 2, 3, \cdots$$

converge or diverge as $n \to +\infty$? Justify your answer and calculate the limit if it converges.

ANSWER: converges to 2

$$\lim_{n\to\infty} \frac{2n+1}{n} = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} = 2.$$

(**b**) Does the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$ converge or diverge? Justify your answer and calculate the limit if it converges.

ANSWER: Diverges.

Since $\lim_{n\to\infty} a_n = 2 \neq 0$, the necessary condition that $\lim_{n\to\infty} a_n = 0$ for convergence fails.

You can also do the comparison test: $\frac{2n+1}{n} > \frac{1}{n}$ and $\sum_{i=1}^{n} \frac{1}{n}$ diverges.

2. (12 points)

(**a**) Find an expression for the general term of the series $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$.

ANSWER: $\frac{(-1)^n}{2n+1} x^{2n+1}$

By inspection.

(**b**) Calculate the radius of convergence.

ANSWER: $R = 1$.

Use the ratio test and calculate

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{1}{2(n+1)+1} x^{2(n+1)+1}\right|}{\left|\frac{1}{2n+1} x^{2n+1}\right|} = \frac{\frac{1}{2n+3}|x|^{2n+3}}{\frac{1}{2n+1}|x|^{2n+1}} = \frac{2n+1}{2n+3}|x|^2$$

and

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{2 + \frac{3}{n}}|x|^2 = \frac{2}{2}|x|^2 = |x|^2$$

Thus, the series converges for $|x|^2 < 1$ – which is equivalent to $|x| < 1$ – and it diverges for $|x|^2 > 1$ – which is equivalent to $|x| > 1$.

3. (14 points) Does the series $\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$ converge or diverge? Justify your answer.

ANSWER: Converges

Since $f(n) = \frac{3}{(2n-1)^2}$ is positive and decreasing for $n \geq 1$, we can use the Integral Test by studying the improper integral $\int_1^{\infty} \frac{3}{(2x-1)^2} dx$. The substitution $w = 2x - 1$ (and hence $dw = 2dx$) shows $\int_1^{\infty} \frac{3}{(2x-1)^2} dx = \frac{3}{2} \int_1^{\infty} \frac{1}{w^2} dw$, which is a convergent integral. The Integral Test implies the series is convergent.

You can use the comparison test: $\frac{3}{(2n-1)^2} \leq \frac{3}{(2n-n)^2} = \frac{3}{n^2}$ and $\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$).

4. (14 points) Does the series $\sum_{n=3}^{\infty} \frac{n+1}{n^2+2n+2}$ converge or diverge? Justify your answer.

ANSWER: Diverges

Since $\frac{n+1}{n^2+2n+2}$ behaves, for large n, like $\frac{n}{n^2} = \frac{1}{n}$ (which defines a divergent p-series), we conjecture this series diverges. To prove this conjecture true by means of the Comparison Test we look for a *lower* bound that defines a divergent series:

$$\frac{n+1}{n^2+2n+2} > \frac{n}{n^2+2n^2+2n^2} = \frac{1}{5}\frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{5}\frac{1}{n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p \geq 1$), the Comparison Test implies the original series also diverges.

5. (12 points) Let a_n be the number of fish living in an alpine lake. Because of environmental changes the population has become endangered to the extent that its numbers decrease 50% each year. Suppose this year we find that there are 1,000 fish in the lake and, in an effort to maintain the population, we decide – starting next year – to add s fish to the lake every year. Then the fish population numbers a_n satisfy the recursion formula

$$a_{n+1} = \frac{1}{2} a_n + s, \qquad n = 0, 1, 2, \cdots.$$

with $a_0 = 1$ (measured in units of a thousand). Here n denotes years in the future ($n = 0$ is this year).

(**a**) Find a formula for a_n in terms of s and n. Do not use sigma summation notation. (HINT: Write out some terms in the sequence, look for a pattern, and think about geometric sequences.)

ANSWER: $a_n = (\frac{1}{2})^n + 2s\left(1 - (\frac{1}{2})^n\right)$

From the first few iterations

$$a_1 = \frac{1}{2} + s$$

$$a_2 = \frac{1}{2} a_1 + s = \left(\frac{1}{2}\right)^2 + \frac{1}{2} s + s$$

$$a_3 = \frac{1}{2} a_2 + d = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 s + \frac{1}{2} s + s$$

$$\vdots$$

we obsere the pattern $a_n = (\frac{1}{2})^n + \left[s(\frac{1}{2})^{n-1} + \cdots + s(\frac{1}{2}) + s\right]$. The expression in the brackets is a geometric sum which we can sum up:

$$a_n = \left(\frac{1}{2}\right)^n + \frac{s\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \left(\frac{1}{2}\right)}$$

(**b**) To what does the number of fish converge as $n \to \infty$?

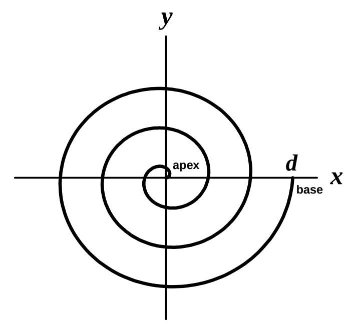
ANSWER: $2s$

$$\lim_{t\to\infty}\left[\left(\frac{1}{2}\right)^n + 2s\left(1 - \left(\frac{1}{2}\right)^n\right)\right] = 0 + 2s(1-0) = 2s.$$

6. (12 points) The human cochlea (inner ear) is a spiral-shaped tube that acts as a frequency analyzer. High frequencies are encoded near the base while low frequencies near the apex (i.e., the center of the spiral as shown in the figure). In polar coordinates, a spiral can be described by the equation $r = c\theta$, where $c > 0$ is a constant. Suppose the human cochlea is a spiral with three turns and has a total length of 35*mm*. What is the direct distance d ("as the crow flies") between the base and apex in *mm*?

(HINT: Set this question up as an arc length problem (in polar coordinates). This will allow you to determine a numerical value for c. You will need to express the three turns of the spiral in terms of radians. Use the integral tables. You may, but you need not, give a decimal approximation to your answer.)

ANSWER: $d = 6\pi\frac{70}{6\pi\sqrt{1+36\pi^2} + \ln\left|6\pi + \sqrt{1+36\pi^2}\right|} \approx 3.671\ mm$



From $x = r\cos\theta$ and $y = r\sin\theta$ we obtain $x = c\theta\cos\theta$ and $y = c\theta\sin\theta$. The distance d is the value of x when $\theta = 6\pi$, i.e., $d = 6\pi c\cos 6\pi = 6\pi c$. We need to determine the value of c from the given fact that the length of the spiral is 35 *mm*. From the arc length formula in polar coordinates

$$\int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

we calculate

$$\begin{aligned}
35 &= \int_0^{6\pi} \sqrt{(-c\theta\sin\theta + c\cos\theta)^2 + (c\theta\cos\theta + c\sin\theta)^2}\, d\theta \\
&= \int_0^{6\pi} \sqrt{c^2(\cos^2\theta + \sin^2\theta)(1+\theta^2)}\, d\theta \\
&= c\int_0^{6\pi} \sqrt{1+\theta^2}\, d\theta = c\frac{1}{2}\left[\theta\sqrt{1+\theta^2} + \ln\left|\theta + \sqrt{1+\theta^2}\right|\right]_{\theta=0}^{\theta=6\pi} \\
&= c\frac{1}{2}\left[6\pi\sqrt{1+36\pi^2} + \ln\left(6\pi + \sqrt{1+36\pi^2}\right)\right]
\end{aligned}$$

and therefore

$$c = \frac{70}{6\pi\sqrt{1+36\pi^2} + \ln\left|6\pi + \sqrt{1+36\pi^2}\right|}.$$

The distance $d = 6\pi c$ is

$$d = 6\pi\frac{70}{6\pi\sqrt{1+36\pi^2} + \ln\left|6\pi + \sqrt{1+36\pi^2}\right|} \approx 3.671mm$$

7. (12 points) Show that the radius of convergence for the binomial series

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

is $R = 1$. (HINT: apply the ratio test.)

ANSWER: The general coefficient is $c_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$ and ratio of successive coefficients

$$\frac{|c_{n+1}|}{|c_n|} = \frac{\left|\frac{p(p-1)(p-2)\cdots(p-(n+1)+1)}{(n+1)!}\right|}{\left|\frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\right|}$$

$$= \frac{|p(p-1)(p-2)\cdots(p-n)|}{(n+1)!}\frac{n!}{|p(p-1)(p-2)\cdots(p-n+1)|} = \frac{|p-n|}{n+1}.$$

The radius of convergence is the reciprocal of the limit

$$\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty}\frac{|p-n|}{n+1} = \lim_{n\to\infty}\frac{\left|\frac{p}{n}-1\right|}{1+\frac{1}{n}} = \frac{|0-1|}{1+0} = 1.$$

8. (12 points)

(**a**) Find the Taylor series for e^{-x^2} at $x = 0$. Write your answer in sigma summation notation. What is the radius of convergence? Show your work. (HINT: use one of the shortcut methods discussed in class, rather than the differentiation formula for the power series coefficients.)

ANSWER: $\sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n x^{2n}$ has infinite radius of convergence $R = +\infty$.

Substitute $-x^2$ for x into the series $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$ to obtain $e^{-x^2} = \sum_{n=0}^{\infty}\frac{1}{n!}(-x^2)^n$. Since the power series for e^x has radius of convergence $R = +\infty$, the same is true for this series.

(**b**) Use your answer in (a) to calculate the limit $\lim_{x\to 0}\frac{e^{-x^2}-1}{x^2}$.

ANSWER: $\lim_{x\to 0}\frac{e^{-x^2}-1}{x^2} = -1$

From (a) we calculate

$$\frac{e^{-x^2}-1}{x^2} = \frac{-x^2 + \frac{1}{3!}x^6 - \frac{1}{4!}x^8 + \cdots}{x^2} = -1 + \frac{1}{3!}x^4 - \frac{1}{4!}x^6 + \cdots$$

and so

$$\lim_{x\to 0}\frac{e^{-x^2}-1}{x^2} = \lim_{x\to 0}\left(-1 + \frac{1}{3!}x^4 - \frac{1}{4!}x^6 + \cdots\right) = -1.$$

EXTRA CREDIT

9. (10 points)

(**a**) Use your answer in Problem 8(a) to obtain the power series for the anti-derivative $\int e^{-x^2} dx$.

ANSWER: $\int e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots$

A power series can be integrated term-wise

$$\int e^{-x^2} dx = \int \left(\sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n x^{2n} \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \int x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

(**b**) Use your answer in (a) to obtain a series for the definite integral $\int_0^1 e^{-x^2} dx$.

ANSWER: $\sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$

By the Fundamental Theorem of Calculus we evaluate the answer in (a) at $x = 1$ and $x = 0$ and then subtract the results:

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2n+1} x^{2n+1} \Bigg|_{x=0}^{x=1} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2n+1} \cdot 1^{2n+1} - 0 = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{2n+1}$$

(**c**) Your answer in (b) should be an alternating series. Use its first two (nonzero) terms to obtain a numerical estimate to the definite integral and then give an error bound for the estimate. Write your answer in the form: *estimate* ± *error bound*.

ANSWER: $\int_0^1 e^{-x^2} dx = \frac{2}{3} \pm \frac{1}{10}$

An error bound for a partial sum of an alternative series is the next term in the series. Thus

$$\left| \int_0^1 e^{-x^2} dx - \left(1 - \frac{1}{3}\right) \right| \le \frac{1}{10}$$