# Collection of Habits of Mind Problems
## University of Nebraska-Lincoln
## Math in the Middle Institute Partnership

**<u>Crazy Eights</u>**

How many ways are there to use 8s strung together with plus (+) signs to equal 1000? Once you think you have an answer, **can you provide a mathematical argument that you are correct**?

<u>Comment</u>: Here is **one** way to do it. You are asked to find how many ways there are to do it.

888+8+8+8+8+8+8+8+8+8+8+8+8+8+8 = 1000

**<u>Coins</u>**

What is the fewest number of coins that it will take to make exactly 43 cents if you have available pennies, nickels, dimes, and quarters? After you have solved this problem, <u>provide an explanation that proves that your answer is correct</u>.

How does the answer (and the justification) change if you only have pennies, dimes, and quarters available? Again, <u>explain why your answer is correct</u>.

**<u>Crossing the River</u>**

A group of adults (teachers and parents) go on a camping trip with a group of 4th grade students. On the first day, the campers (adults and students) come to a river. It's not a very wide river, but it is too deep to wade across. The campers find a canoe, but the canoe is not very big. Unfortunately, the adults are rather big and only one adult can fit in the canoe at one time. Fortunately, the 4th grade students are small enough that the canoe will hold any two of the students. It is not possible for one adult and one student to fit in the canoe at the same time. However the students have experience with canoes and each can safely row across the river by themselves.

- Suppose that there are four adults and two students on the camping trip. Is it possible to get the entire camping group across the river? If yes, how many one-way trips across the river will it take to get all six people across the river?
- What if there were 5 adults and only one student on the trip? Is it possible to get the entire group across the river? If yes, how many one-way trips will it take?
- If there were 5 adults and 2 students, how many one-way trips will it take?
- If there were 4 adults and 6 students, how many one-way trips will it take?
- How can this problem be generalized? That is, how can you solve the problem for any number of adults and any number of students?

**<u>Dodge Ball</u>**

Dodge Ball is a game for two players – Player One and Player Two. Each player has her own special board and is given six turns. Here is a copy of each player's board.

**Player One's game board:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |

**Player Two's game board:**

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| | | | | | |

Player One begins by filling in the first horizontal row of her table with a run of X's and O's. That is, on the first line of her board, she will write six letters – one in each box – each letter being either an X or an O. Player Two looks at Player One's first row and then places either one X or one O in the first box of her board. So at this point, Player One has filled in the first row of her board with six letters, and Player Two has filled in the first box of her board with one letter.

The game continues with Player One noticing what Player Two placed on her board and then writing down a run of six letters (X's and O's), one in each box of the second horizontal row of her board, followed by Player Two writing one letter (an X or an O) in the second box of her board. This game proceeds in this fashion, with each player's moves visible to the other, until all of Player One's boxes are filled with X's and O's; thus, Player One has produced six rows of six marks each, and Player Two has produced one row of six marks. Player One wins if any horizontal row she wrote down is identical to the row that Player Two created. Player Two wins if Player Two's string is not one of the six strings made by Player One.

Would you rather be Player One or Player Two? Who has the advantage? Can you describe a strategy for either side that will always result in victory? Give a careful justification of an position that you take.

## Triangular Numbers

Carl Friedrich Gauss is one of the greatest mathematicians of all time. Legend has it that when he was in the third grade his teacher told his class to add the whole numbers from 1 through 100 and "I don't want to see anyone's eyes until you are finished." Instantly, Carl raised his hand. After a scolding from the teacher for not following the rules, Carl said, "But I have the answer; it is 5050." How did Carl get the answer so quickly?

Numbers that you get by adding consecutive whole numbers starting with 1 are called triangular numbers.

For example, 1, 3, and 6 are triangular numbers as the following computations show:

$1 = 1$
$3 = 1 + 2$
$6 = 1 + 2 + 3$
...

1. Continue the above list until you have the list of the first 15 triangular numbers. Suppose that we wanted to know the 50th triangular number. It would be tedious to list all the triangular numbers up to that point. Find a formula for the 50th, 100th, and nth triangular numbers and justify why the formulas make sense.

2. Draw a circle. Place 8 dots fairly evenly around the circle. Connect each pair of dots by a line segment. We call these segments chords. How many distinct chords do you get? (Careful, if you used only 4 dots, then you would get 6 chords.) Find a formula for the number of chords with 50, 100, or n dots and justify why the formulas make sense.

3. Now suppose you have 15 students in your class, and you want to pick two students to be "Class Leaders" for the day. You wonder, "How many different pairings are possible?" ("Alex and Claire" and "Alex and Tess" would be two different pairings, but "Alex and Claire" would be the same pair as "Claire and Alex.") Find a formula for the possible choices for Class Leaders if the class has 50, 100, or n students and justify why the formulas make sense.

4. There are relationships between the first three parts of this problem. Describe the relationships.

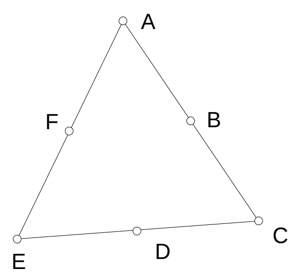
5. What is an explanation as to how young Carl may have figured out the sum of the first 100 whole numbers so quickly?

All explanations should be complete and enable the rest of us to understand the process.

## The Triangle Game

Consider an equilateral triangle with points located at each vertex and at each midpoint of a side. (See picture below.) This problem uses the set of numbers {1, 2, 3, 4, 5, 6}.

1) Find a way to put one of the numbers at each point so that the sum of the numbers along any side (two vertices and one midpoint) is equal to the sum of the numbers along each of the two other sides. (Call this number the "Side Sum.")
2) Is there more than one way to get the same Side Sum? Justify your answer. You may have to make a decision as to what it means for two "ways" to be different.
3) It is possible to have different Side Sums. What is the smallest possible Side Sum and why? What is the largest possible Side Sum and why?
4) What Side Sums are possible? Show by giving examples for each Side Sum.
5) What is a possible generalization of The Triangle Game to other polygons? Find some solutions by giving examples to your generalized game.




## Mind over Mathematics?

On this assignment, I will demonstrate my ability to read your mind.

Choose a six-digit whole number "n" that repeats the first three digits – e.g. numbers like 725725 or 109109 or 226226. Write your number in the following space:

My number is: n = ________________________.

Without knowing what number you choose, I will guess a factor of your number (i.e. a number "d" chosen so that n/d is a whole number. (I will not guess the number 1 and I will not guess the number n.) Just before you turn in your homework, I will give you my choice for d. Please write it here:

d = ________________________.

Your assignment is to expose my mind reading scam as just good mathematics. On the line below, predict the number that I will choose as a factor of the number you choose for n. You are permitted a maximum of 7 different choices. Then, explain my "mind-reading" trick.

_______ _______ ________ ________ ________ ________ ________

## A Shaky Story

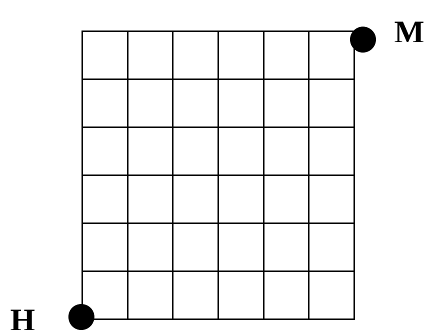
Stacy and Sam Smyth were known for throwing good parties. At one of their gatherings, five couples were present (including the Smyths). The attendees were cordial, and some even shook hands with other guests. Although we have no idea who shook hands with whom, we do know that no one shook hands with themselves and no one shook hands with his or her own spouse. Given these facts, a guest might shake hands with as many as eight other people or the guest might not shake anyone's hands.

At midnight, Sam Smyth gathered the crowd together and asked the other nine people how many hands each of them had shaken. Much to Sam's amazement, each person gave a different answer. That is, someone didn't shake any hands, someone else shook one hand, someone shook two hands, someone else shook three hands, and so forth, down to the last person who shook eight hands.

Given this information, determine the exact number of hands that Stacy Smyth shook. (Explain your answer, of course.)

## Mathville

Downtown Mathville is laid out as a 6 X 6 square grid of streets (See diagram). Your apartment is located at the Southwest corner of downtown Mathville. (See point H.) Your math classroom is located at the Northwest corner of downtown Mathville. (See point M.) You know that it is a 12 block walk to math class and that there is no shorter path. Your curious roommate (we'll call her Curious Georgia) asks how many different paths (of length 12 blocks – you don't want backtrack or go out of your way) could you take to get from your apartment to the math class. It should also be clear that no shorter path exists. Can you solve Curious Georgia's math problem?



## The Case of the Missing Gradebook

Andy, Barb, Cindy and Don are four of Mrs. Jones' most mischievous students. While Mrs. Jones was teaching a lesson on fractions, all four students said that they needed to sharpen their pencil. (The pencil sharpener is right beside Mrs. Jones' desk.) Mrs. Jones said they could sharpen their pencils but that they would have to go up to her desk one at a time.

Later after the lesson had ended, Mrs. Jones noticed that her gradebook was missing from her desk. Suspecting that one of her four mischievous students must have taken the grade book, she interviewed them one at a time. Being quite mischievous, all four told Mrs. Jones two lies. Here is what they said:

Andy

- None of us took the gradebook.
- The gradebook was on your desk when I returned to my seat.

Barb

- I was the second person to sharpen my pencil.
- The gradebook was already missing when I went to sharpen my pencil.

Cindy

- I was the third person to sharpen my pencil.
- The gradebook was on your desk when I returned to my seat.

Don

- The person who took the gradebook did not sharpen their pencil after me.
- The gradebook was already missing when I went to sharpen my pencil.

Mrs. Jones does not remember the order in which the students went to sharpen their pencils.

Use your deductive powers to determine who took the gradebook. Be sure to provide a careful explanation for your answer.

## An Open and Shut Case

1) In a certain school there are 100 lockers lining a long hallway. The lockers are numbered 1, 2, 3,..., 99, 100. **All are closed**. Suppose that 100 students walk down the hall in single file, one after another. Suppose the first student (who we will call "Student #1" for obvious reasons) opens every locker. The second student (i.e. Student #2) comes along and closes every 2$^{nd}$ locker beginning with locker #2. (i.e. lockers #2, 4, 6,…, 98, 100). Along comes Student #3 who **changes** the position of every third locker; if it is open, this student closes it; if it is closed, this student opens it. (I.e., (s)he closes locker #3, opens locker #6, closes locker #9, etc.) Student #4 changes the "open or shut" position of every fourth locker, and so forth, until the 100$^{th}$ student changes the position of locker #100.

**Which lockers are open at the end of this event?**

2) Can you extend this? I.e., if in the problem above, we had 1,000 (or even 10,000) numbered lockers and people, which lockers would be open at the end of the event?

3) Explain **why** these particular numbers are the numbers of the lockers that are open at the end of the event?

## The Chess Game Problem

There once was a humble servant who was also a chess master. He taught his king to play the game of chess. The king became fascinated by the game and offered the servant gold or jewels in payment, but the servant replied that he only wanted rice—one grain for the first square of the chess board, two on the second, four on the third, and so on with each square receiving twice as much as the previous square. The king quickly agreed. How much rice does the king owe the chess master? Suppose it was your job to pick up the rice. What might you use to collect the rice, a grocery sack, a wheelbarrow, or perhaps a Mac truck? Where might you store the rice?

## An (un)common Solution

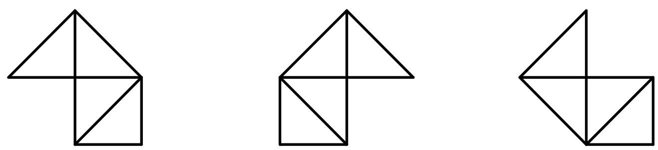
Find a positive integer which if divided by 2 leaves a remainder of 1, divided by 3 leaves a remainder of 2, divided by 4 leaves a remainder of 3, divided by 5 leaves a remainder of 4, divided by 6 leaves a remainder of 5, divided by 7 leaves a remainder of 6, divided by 8 leaves a remainder of 7, and divided by 9 leaves a remainder of 8. Hint: Remember Polya's problem solving rules. It helps to try a simpler problem. *Extension:* Try to find more than one solution (an infinite number?) and by trying to find the smallest positive integer that is a solution.

**<u>Shapes from Four Triangles</u>** (Adapted from Shapes and Measurement, written by the Reconceptualizing Mathematics Project at San Diego State University.)

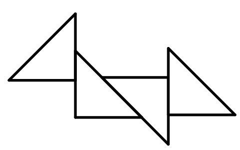
Given four congruent isosceles right triangles, how many different polygonal regions can you make, using all four triangles each time? Two shapes are said to be the same if some combination of a slide, a rotation, or a flip will transform the first shape into the second shape. To answer the question of "how many?" you will probably want to find and display all possible Shapes from Four Triangles. Once you have done this, your task is to find an argument that **proves** you have found a complete set of Shapes from Four Triangles. I.e., how would you explain to someone who isn't looking at the shapes that

i) your shapes are all different, and
ii) there are no more left to be found?

Note: The following three are all the SAME polygonal region (perhaps called square-with-sail), since some rigid motion shows they are congruent and hence they are not really different shapes.



If it is not clear, the intent is that a side of one triangle should fit exactly on a congruent side of an attached triangle. Making a shape involving something like the following is <u>not</u> allowed:



**Star Polygons**

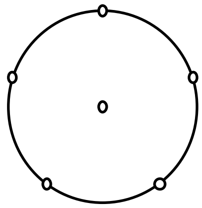


Fig. 1

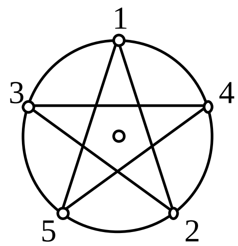


Fig. 2

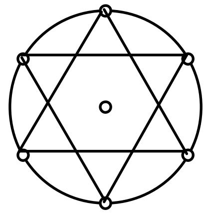


Fig. 3

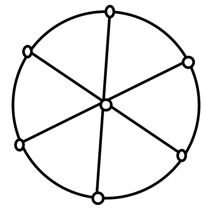


Fig. 4

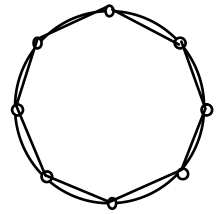


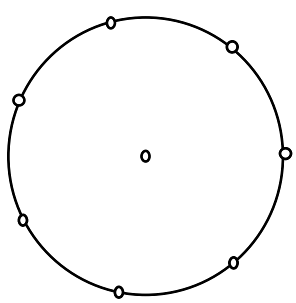
Fig. 5

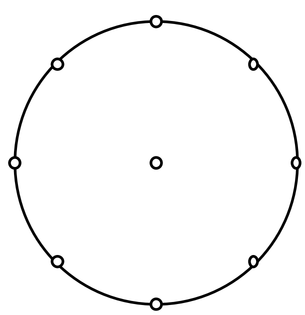
In Figure 1, the five points on the circle are evenly spaced. If you start at any of the points and skip to the second (clockwise) point, and from there skip to the second (clockwise) point, etc. (e.g., in Figure 2, 1-->2-->3-->4-->5-->1), *without lifting your pencil* you reach every point and get the star shape in Figure 2, a regular **star polygon**, 5-pointed here although that is <u>not</u> a requirement to be a star polygon. If you skip to every third point (clockwise) it is the same as skipping to the second (counterclockwise) point, so the resulting star polygon is not really a new design.

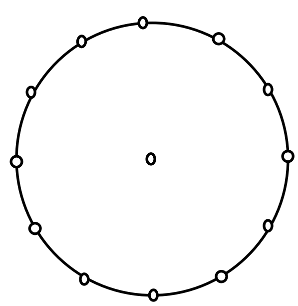
If you start with six evenly spaced points (Fig. 3), however, and join every second point, the resulting shape requires **lifting your pencil** to reach every point. So even though Fig. 3 looks good, it is **not** considered a <u>single</u> polygon and so is not a star polygon. If you joined every third point (Fig. 4), you get only the 3 line segments and not a star polygon. If you skip to the first point each time, as in Fig. 5, you get a polygon but not a <u>star</u> polygon. (If the points are not equally spaced, any star polygon obtained is not a <u>regular</u> star polygon, but it is still a star polygon.)

**The Problem**: Explore the method above to see whether getting a regular n-pointed star polygon (without lifting your pencil) is predictable. That is, if you are given a certain number <u>n</u> of equally spaced points on a circle and the number <u>s</u> indicating how to skip, you will be able to tell whether or not a regular n-pointed star polygon will result. For examples, in Fig. 2, n = 5 and s = 2 (s = 2 means you skip to the 2nd point each time) give a regular star-polygon, but in the other two, you do <u>not</u> get regular star-polygons: in Fig. 3, n = 6 and s = 2; in Fig. 4, n = 6 and s = 3; in Fig. 5, n = 8 and s = 1. A complete solution to the problem would enable you to predict what happens when someone gives you <u>any</u> values for n and s. That might be too much to figure out during the exam. But can you find an answer for a few cases like n = 21 and s = 5, or n = 35 and s = 14?

Here are some sample circles with evenly spaced points, for modeling your own circles. You may want to try other values for n, of course.







## Number Triangle

Create a number triangle by listing consecutive odd numbers as shown below, with each row having one more number than the preceding one:

Row 1: 1
Row 2: 3 5
Row 3: 7 9 11
Row 4: 13 15 17 19
Row 5: 21 23 25 27 29

**a)** What is the first number in Row 20? What is the last number in Row 20?
**b)** Determine the sum of all the numbers in each row. What do you think is the sum of all the numbers in Row *n*?
**c)** Using the information from part 2, find the mean of the entries in each row. What do you conjecture is the mean of Row 20?
**d)** Compare the row means with the number of each row and the first number in each row. Describe the pattern that you see. From this pattern, can you guess a formula for the first number in each row?
**e)** Explain why your guess is correct.

## Looking Squarely at a Checkerboard

1. Two red (i.e. non-black) squares are removed from opposite corners of a checkerboard. Given a set of 31 dominoes each the size of two squares, is it possible to cover the remaining 62 squares with the 31 dominoes? Explain.

2. How many different squares are there on an 8x8 checkerboard? If we double the number of rows and columns of the checkerboard, how does the number of squares change?

**The Box Design Log Problem**

July 25, 2006

Dear Math in the Middle Teachers:

$M^2$ plans to market boxes that hold exactly 200 one-inch cubes. Your assignment is:

(1) Find all possible dimensions for boxes that will hold exactly 200 one-inch cubes. Do not worry about where the top of the box is; we will decide that. Your report should list all possible dimensions and, should explain how you know that you have found all the possibilities.

(2) Our Marketing Department has come up with an eye-catching foil covering for the box. The covering material is, however, quite expensive and so we would like to minimize the cost of the covering material. Which choice for the dimensions of the box will minimize the total cost of the covering material? The cost of the cardboard box itself will be negligible compared to that of the covering material and the cubes.

(3) Finally, we want your personal recommendation for the best dimensions to use. We realize that there may be factors other than the cost of covering and will appreciate getting your views on this issue. Your reasons for your personal recommendation are important to us.

We have put this project on the "fast track" and must have your report tomorrow, July 26, 2006.

Sincerely,

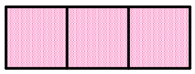
Jim Lewis
Chief Financial Officer, $M^2$ Unlimited, Inc.

CC: Cheryl Olsen
Patience Fisher

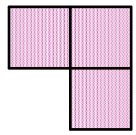
## Area and Perimeter

**Is there a relationship between the area and the perimeter of a polygonal shape made with congruent square regions?** Note: Squares must be joined complete-side to complete-side. The outside "boundary" should be a polygon. In particular, this would not permit a shape with a "hole" in the middle. You may work on this problem in groups and you may seek help from your instructors. You may not discuss this problem with anyone else.

**Example**: These two shapes have the same perimeter (8 units) and the same area (3 units).



and



A. Find examples to show that different shapes can have the same area but have different perimeters. Similarly, show that different shapes can have the same perimeters but can have different areas. (Hint: Try four square regions.)

B. Make a table with column headings like the following, gather data, and look for relationships.

| Area | Perimeter Minimum | Perimeter Maximum |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| etc. | | |

C. Make a similar table, but keep the perimeter the same in each line of the table. Again, look for relationships.

| Perimeter | Area Minimum | Area Maximum |
|---|---|---|
| 4 | | |
| 6 | | |
| 8 | | |
| 10 | | |
| 12 | | |
| 14 | | |
| etc. | | |

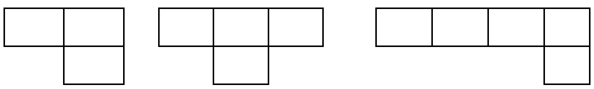
D. Write a report summarizing your findings. Try to give an argument to justify any formulae you discover.

## Pentominos

**Background:** A domino can be thought of as two squares joined along one side. Similarly, a triomino might be a polygon formed by joining three squares together. (In each case two matching sides must fit together exactly.) Continue in this manner; define what is meant by a tetromino (i.e. a polygon created by joining four squares together) and a pentomino (a polygon created by joining five squares together). We will say that two pentominos are the same if one can be shifted, rotated and/or flipped to fit exactly onto the other pentomino.

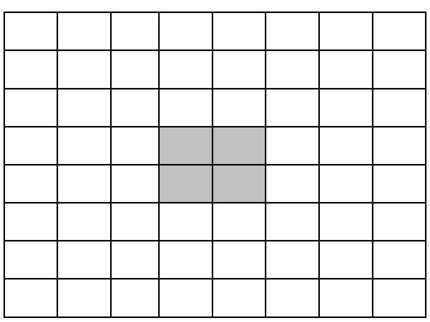
**Examples:**

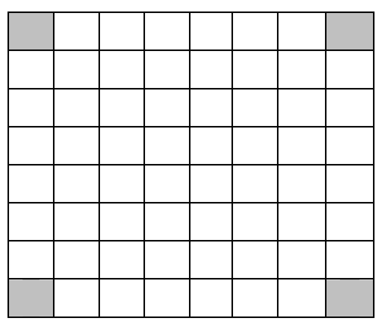
Triomino Tetromino Pentomino



**Warm-up:** How many pentominos are there? Many of you will know the answer to this. In fact, you may have a set of pentominos in your classroom. Just to level the playing field, we will give the answer. There are 12 pentominos.

**A Fun Activity**: Since there are 12 pentominos, each with five squares, it is conceivable that one can place the pentominos on a checkerboard (with the same size squares) without any overlap, thus leaving four squares "uncovered." Can you do it? Once you find one way to do it, you might take on the challenge of finding a way to cover the checkerboard leaving a prescribed set of four squares uncovered. For those willing to accept the challenge, here are two designs you might try to create. (The shaded squares are left uncovered.)





**The Challenge of Explaining Why:**

Once you have convinced yourself that there are exactly 12 different pentominoes, can you provide an argument that your answer is correct? I.e. Can you give a convincing argument that "all 12 are different" and "there are no more?" Note that this calls for more than statements like, "I tried all flips and rotations and no two are the same," or "I considered all the possibilities and couldn't create any new pentominos." Those are "trust me" arguments. What this calls for is a discussion of the attributes of the pentominos that explains why they are different and why there are no more.

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**Coins**
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**An (un)common Solution**
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