Mathematicians Working with Teachers

Examples from

Focus on Mathematics

Al Cuoco
Center for Mathematics Education
EDC
alcuoco@@edc.org
http://www2.edc.org/cme/showcase

An "Inside-Out" Approach to Useful Mathematics for Teaching

1. *Look at mathematics used in the daily work of teaching*

2. *Look at the knowledge used by expert teachers*

3. *Design programs and resources that help teachers develop this expertise*

(1) Look at the daily work of teaching

• The class is using calculators and estimation to get decimal approximations to $\sqrt{5}$. One student looks at how you do out long multiplication and realizes that none of these decimals would ever work, because if you square a finite (non-integer) decimal, there'll be a digit to the right of the decimal point. So you can't ever get an integer. She deduces that that $\sqrt{5}$ can't be rational.

— adapted from "A Dialogue About Teaching" in *What's Happening in Math Class?* Teacher's College Press.

• Nine year old David, experimenting with numbers, conjectures that, if the period for the decimal expansion of $\frac{1}{n}$ is $n-1$, then n is prime.

— Adapted from a Reader Reflection by Walt Levisee in the *Mathematics Teacher* (March, 1997).

• Speaking of decimals, how would you characterize the "unit fractions" $\frac{1}{n}$ that have terminating decimal expansions? What can you say about the periods of the repeating ones?

• This problem causes no difficulty with prealgebra students who understand the connection between rate, time, and distance:

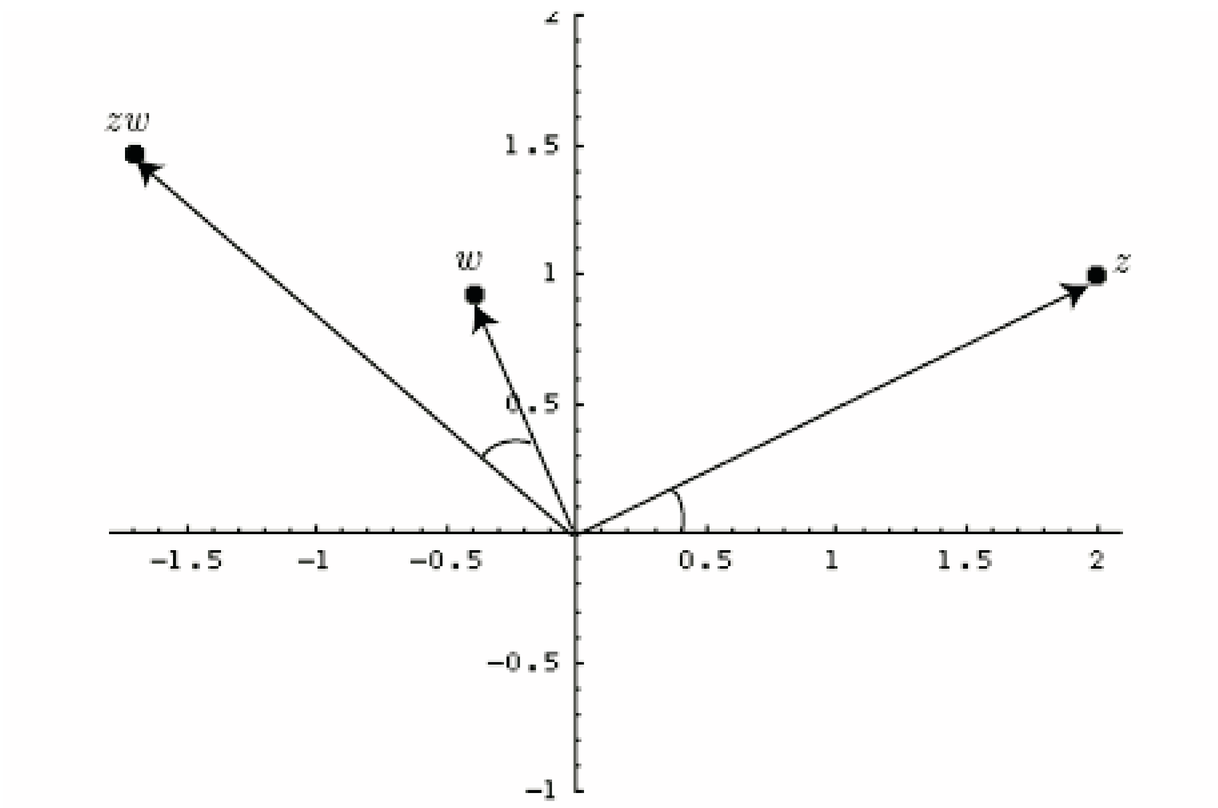
> *Mary drives from Boston to Washington, a trip of 500 miles. If she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back, how many hours does her trip take?*

While *this* problem is baffling to many of the same students a year later in algebra class:

> *Mary drives from Boston to Washington, and she travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes* $18\frac{1}{3}$ *hours, how far is Boston from Washington?*

• How can you help your students understand the "multiplication rule" for complex numbers?

$|zw| = |z|\,|w|$ and $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$



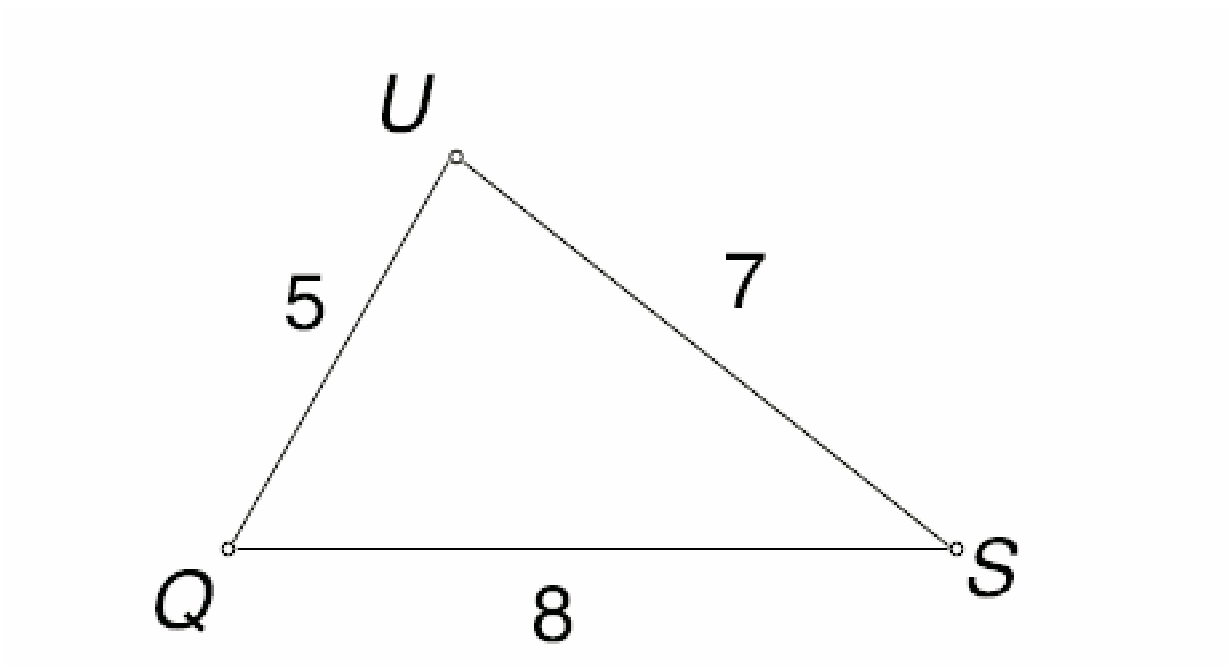
What if they didn't know any trig?

• How can you find a “simple” rule that agrees with this table?

Input	Output
0	1
1	6
2	63
3	364
4	1365
5	3906
6	9331
7	19608

What’s known about this and related problems?

• How can you generate "nice" problems, for example:



How big is $\angle Q$?

• **(How) Can you generate the graph of $y = \sin x$ in Geometer's Sketchpad—without using animation?**

• **(How) Can you use a CAS to get a formula for**

- $\cos 5x$?

- $\sum_{k=0}^{n-1} k^5$?

- The distribution of sums when 4 dice are thrown?

- **All of these contexts are indigenous to teaching.**

- **All of them involve sophisticated content knowledge.**

- **And the list goes on...**

  - *Form versus function*: why is $(x-1)(x+1) = x^2 - 1$?

  - A student says that calculating with complex numbers is the same as calculating with polynomials, except, at the end, you just replace i^2 by -1.

  - What's the story with trig identities?

(2) Look at the knowledge used by expert teachers

They know mathematics as a scholar: They have a solid
grounding in classical mathematics, including

- its major results
- its history of ideas
- its connections to precollege mathematics

They know mathematics as an educator: They understand the habits of mind that underlie major branches of mathematics and how they develop in learners, including

- algebra and arithmetic
- geometry
- analysis

They know mathematics as a mathematician: They have *experienced the doing of mathematics* —they know what it's like to

- grapple with problems
- build abstractions
- develop theories
- become completely absorbed in mathematical activity for a sustained period of time

They know mathematics as a teacher: They are expert in uses of mathematics that are specific to the profession, including

- the ability "to think deeply about simple things" (*Arnold Ross*)

- the craft of task design

- the ability to see underlying themes and connections

- the "mining" of student ideas

Mathematics for Teaching Demands "Knowing":

- ***The "facts"***

- ***The "epistemology"***

- ***The "experience"*** $(*)$

- ***The "craft"*** $(*)$

$(*)$ ***are essential pieces missing in many preservice and professional development programs***

(3) Design programs and resources that help teachers develop this expertise

Example—A Mathematics and Science Partnership: Focus on Mathematics

`http://focusonmath.org`

FoM is a Mathematics-Science Partnership funded by the National Science Foundation. It includes 9 partners:

Focus on Mathematics

- Boston University, UMASS Lowell
- Education Development Center
- Mathematics faculty and administrators from 5 school districts:
  - Arlington Public Schools
  - Chelsea Public Schools
  - Lawrence Public Schools
  - Waltham Public Schools
  - Watertown Public Schools
- Lesley University (Evaluation)

Our Approach

- **Depth over breadth**
  - We work intensely on *one* aspect of improving education
- **Focus on mathematics**
  - Everything we do revolves around mathematics
- **Capacity building**
  - Teachers will drive professional development
- **Community building**
  - Mathematicians, teachers, and education researchers

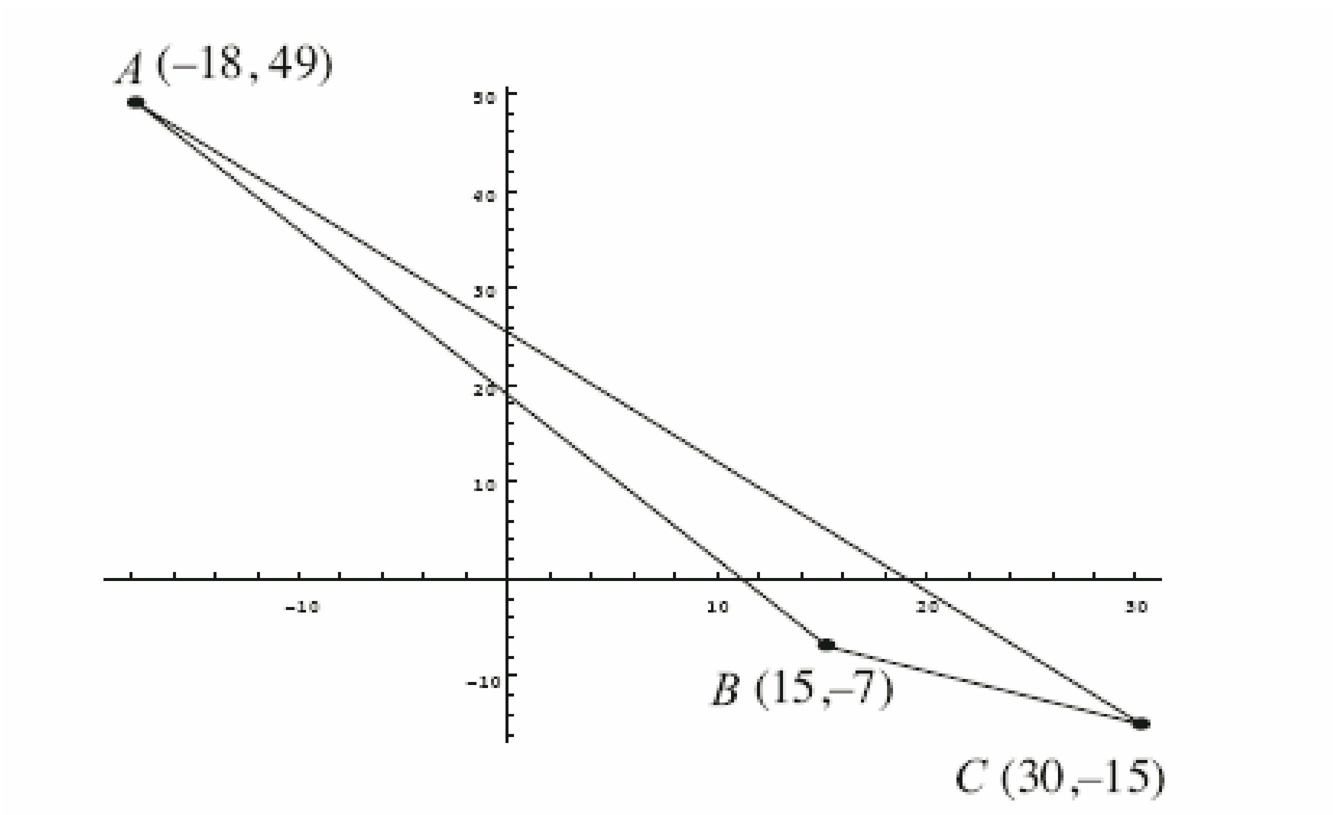
Some of the Programs

- Academic Year Seminars
- Colloquia
- Summer Institutes
- Research Projects
- A New Graduate Program
- Study Groups

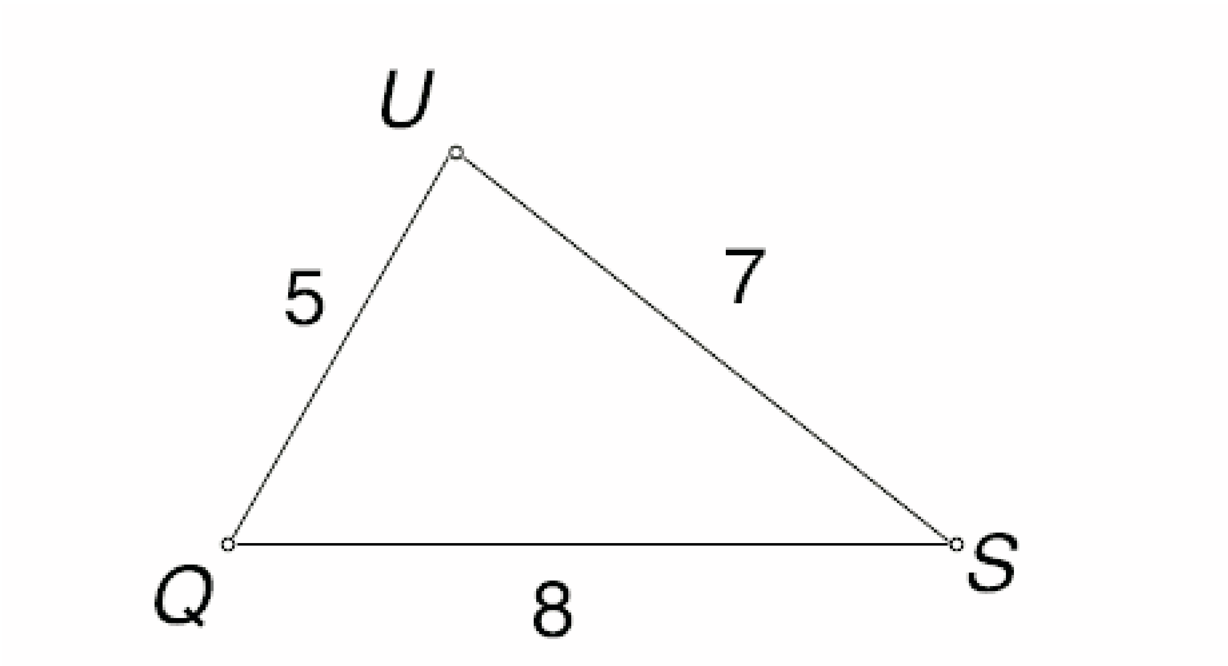
Example: The Mathematics of Task Design
Lawrence High School

The vertices of a triangle have coordinates

$$(-18, 49), (15, -7), (30, -15)$$



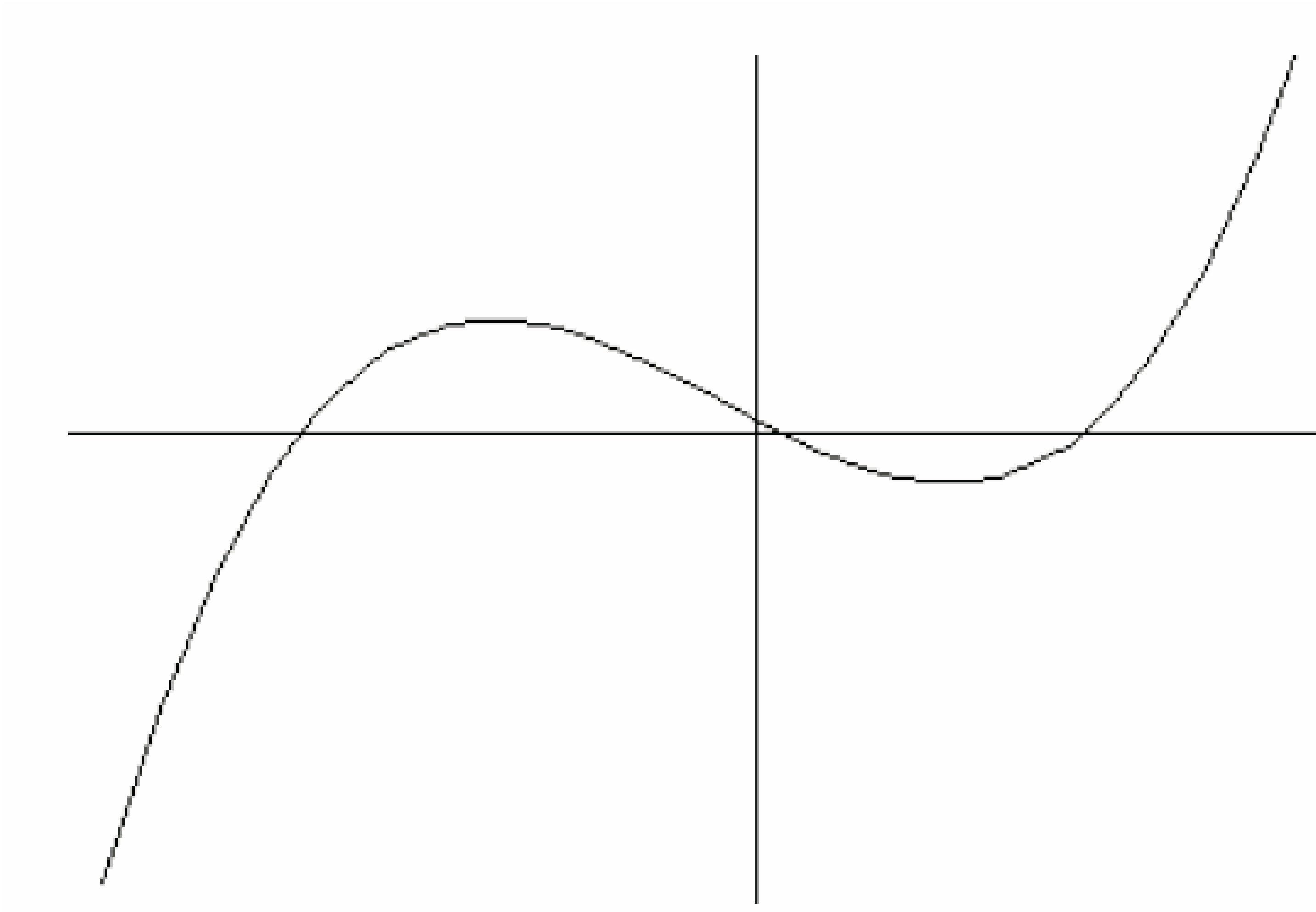
A strange but nice triangle; how long are the sides?



How big is angle Q?

Fold up to make a box

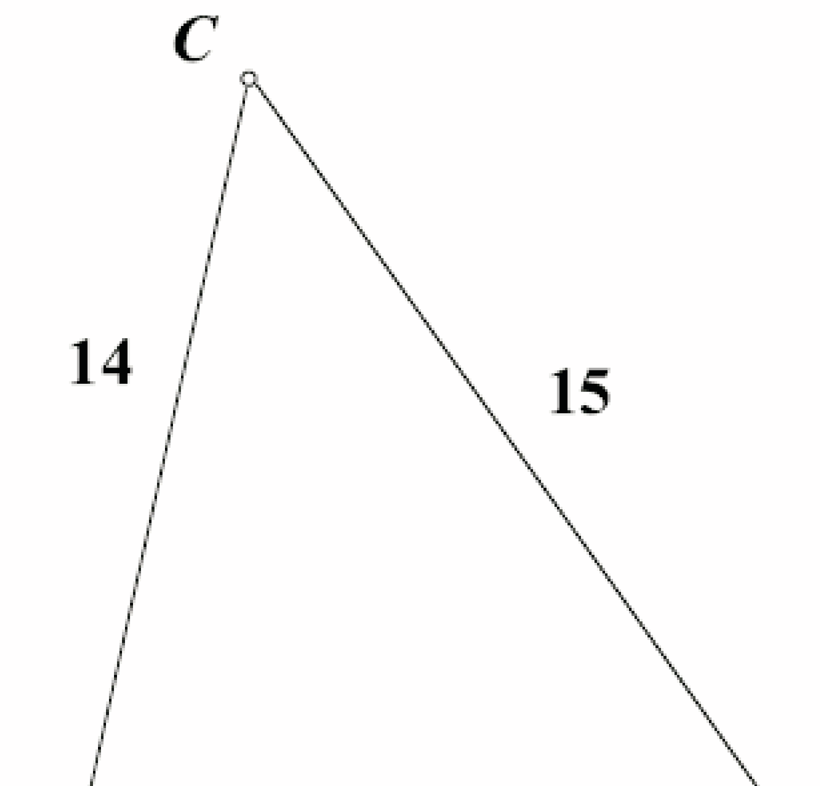
What size cut-out maximizes the volume?



$$f(x) = 140 - 144\,x + 3\,x^2 + x^3$$

Find the zeros, extrema, and inflection points

Find the area of this triangle



An Algebraic Approach

How to Amaze Your Friends at Parties

A *Gaussian Integer* is a complex number of the form $a + bi$ where a and b are *integers*.

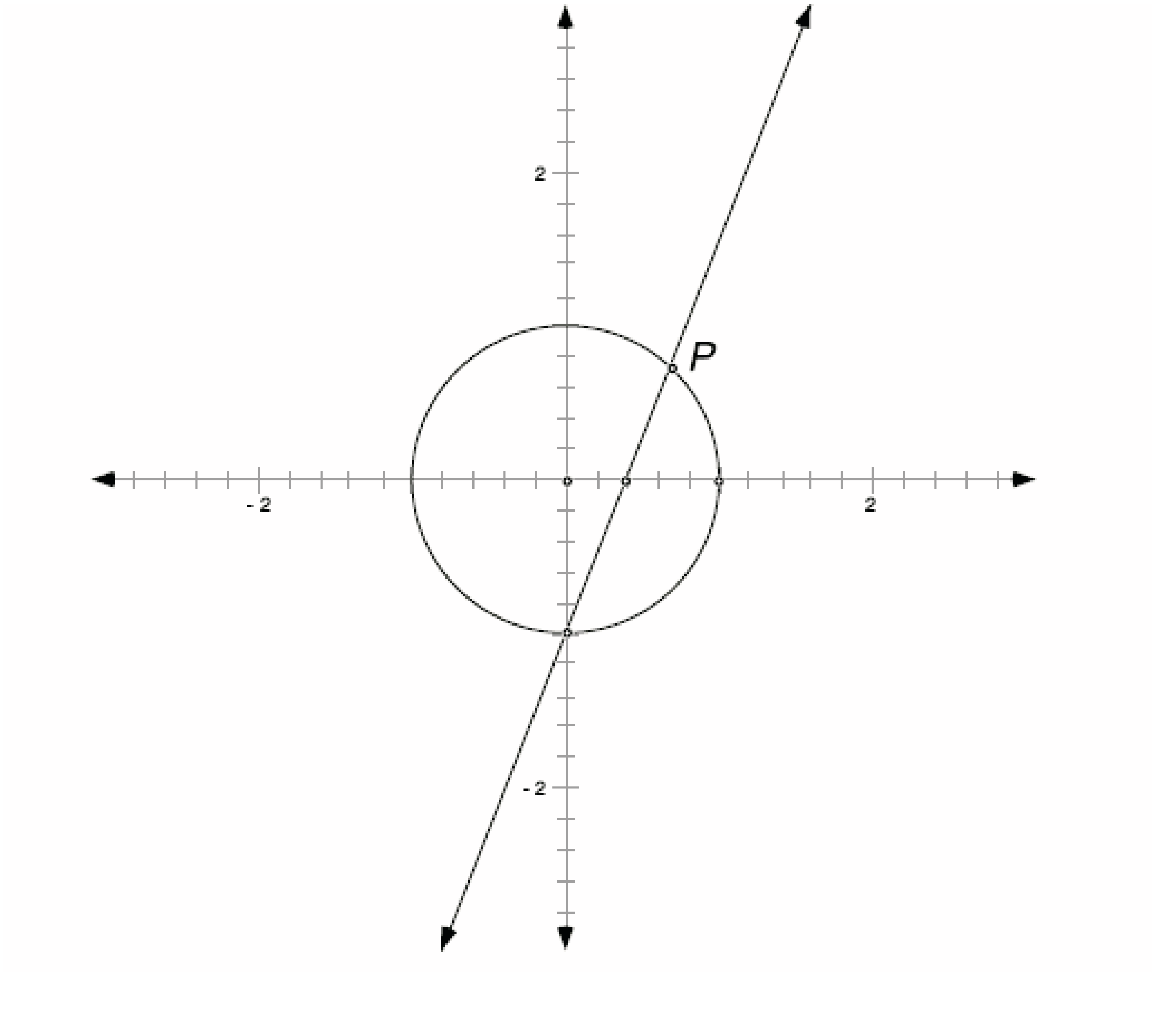
Example: $3 + 2i$.

Non-example: $\frac{1}{2} + i\sqrt{2}$.

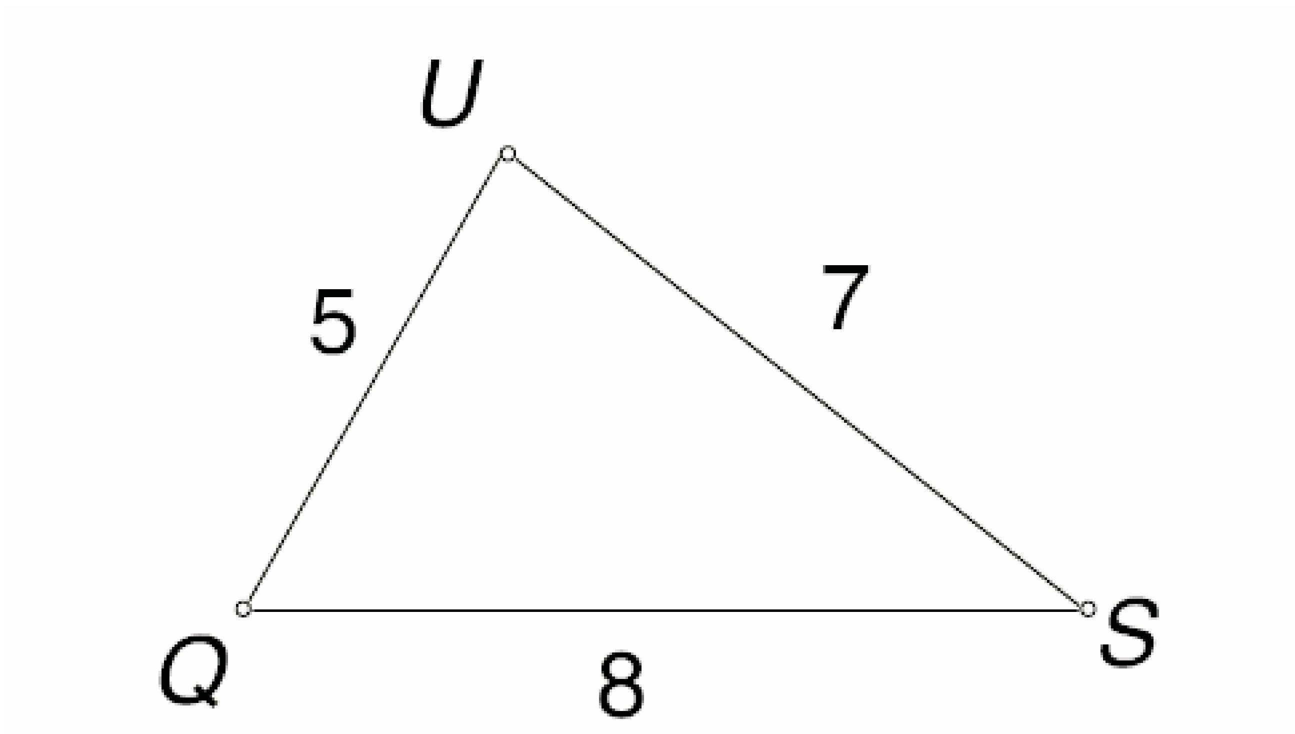
Pick your favorite Gaussian Integer (make $a > b$) and square it.

$s \rightarrow$ $r \downarrow \quad (r+si)^2 \searrow$	1	2	3	4	5	6
2	$3+4i, 5$					
3	$8+6i, 10$	$5+12i, 13$				
4	$15+8i, 17$	$12+16i, 20$	$7+24i, 25$			
5	$24+10i, 26$	$21+20i, 29$	$16+30i, 34$	$9+40i, 41$		
6	$35+12i, 37$	$32+24i, 40$	$27+36i, 45$	$20+48i, 52$	$11+60i, 61$	
7	$48+14i, 50$	$45+28i, 53$	$40+42i, 58$	$33+56i, 65$	$24+70i, 74$	$13+84i, 85$
8	$63+16i, 65$	$60+32i, 68$	$55+48i, 73$	$48+64i, 80$	$39+80i, 89$	$28+96i, 100$

A Geometric Approach



If the line has rational slope, P has rational coordinates



A nice triangle

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos 60^\circ \\ &= a^2 + b^2 - 2ab\,\tfrac{1}{2} \\ &= a^2 - ab + b^2 \end{aligned}$$

An Algebraic Approach

So, we want integers (a, b, c) so that

$$c^2 = a^2 - ab + b^2$$

Call such a triple an "Eisenstein Triple."

Let

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

$[\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0]$

and consider $\mathbf{Z}[\omega] = \{\mathrm{a} + \mathrm{b}\omega \mid \mathrm{a}, \mathrm{b} \in \mathbf{Z}\}$.

Then $(a + b\omega)(a + b\overline{\omega}) = a^2 - ab + b^2$.

Start with $z = 3 + 2\omega$. Square it:

$$\begin{aligned} z^2 &= (3 + 2\omega)^2 \\ &= 9 + 12\omega + 4\omega^2 \\ &= 9 + 12\omega + 4(-1 - \omega) \quad (\omega^2 + \omega + 1 = 0) \\ &= 5 + 8\omega \end{aligned}$$

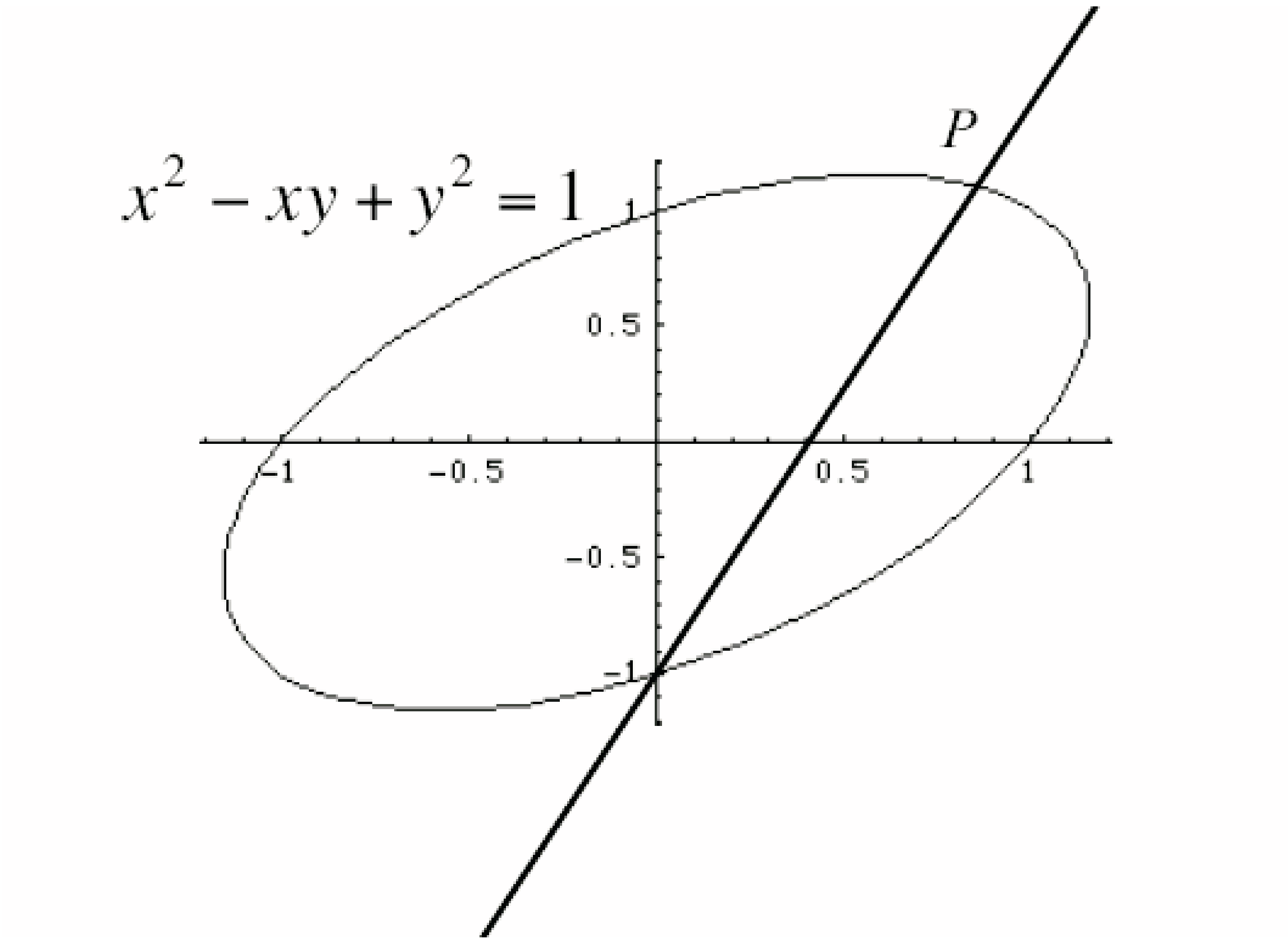
and voilà:

$$5^2 - 5 \cdot 8 + 8^2 = 49, \quad \text{a perfect square.}$$

So the triangle whose sides have length 5, 8, and 7 has a 60° angle.

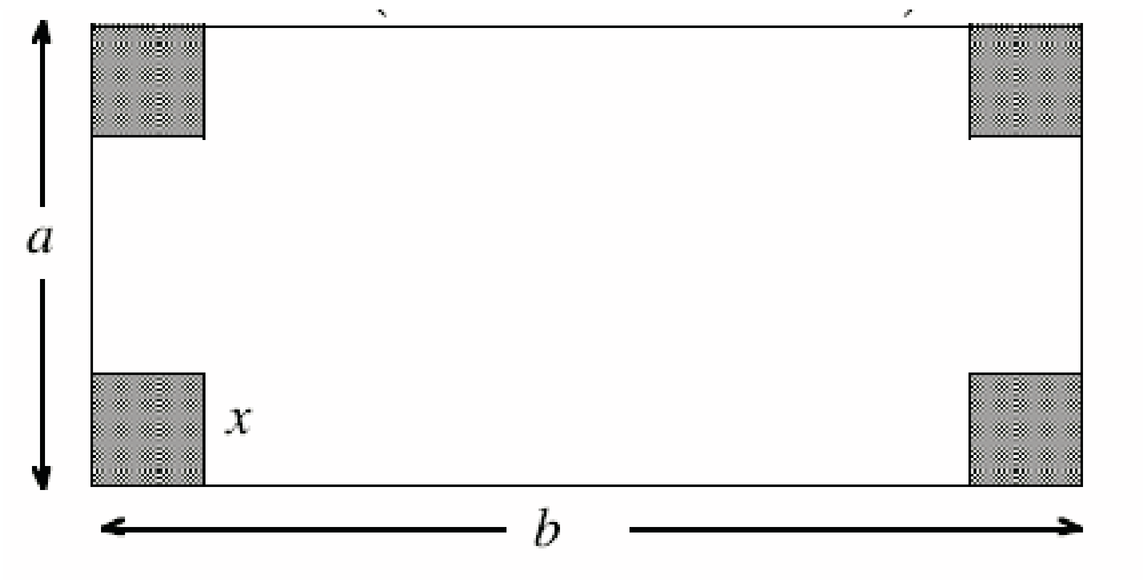
$r \downarrow$ $(r+s\omega)^2 \searrow$ / $s \rightarrow$	1	2	3	4	5
2	$3+3\omega, 3$				
3	$8+5\omega, 7$	$5+8\omega, 7$			
4	$15+7\omega, 13$	$12+12\omega, 12$	$7+15\omega, 13$		
5	$24+9\omega, 21$	$21+16\omega, 19$	$16+21\omega, 19$	$9+24\omega, 21$	
6	$35+11\omega, 31$	$32+20\omega, 28$	$27+27\omega, 27$	$20+32\omega, 28$	$11+35\omega, 31$
7	$48+13\omega, 43$	$45+24\omega, 39$	$40+33\omega, 37$	$33+40\omega, 37$	$24+45\omega, 39$
8	$63+15\omega, 57$	$60+28\omega, 52$	$55+39\omega, 49$	$48+48\omega, 48$	$39+55\omega, 49$
9	$80+17\omega, 73$	$77+32\omega, 67$	$72+45\omega, 63$	$65+56\omega, 61$	$56+65\omega, 61$
10	$99+19\omega, 91$	$96+36\omega, 84$	$91+51\omega, 79$	$84+64\omega, 76$	$75+75\omega, 75$

A Geometric Approach



If the line has rational slope, P has rational coordinates

Reasonable (and Rational) Boxes



Soon to be a box

Well, as we tell our students, let the size of the cut-out be x. Then the volume is a function of x:

$$V(x) = (a - 2x)(b - 2x)x = 4x^3 - 2(a+b)x^2 + abx,$$

so,

$$V'(x) = 12x^2 - 4(a+b)x + ab.$$

We want this to have rational zeros, so we want the discriminant $16(a+b)^2 - 48ab$ to be a perfect square. But 16 is a perfect square, so we want to make

$$(a+b)^2 - 3ab = a^2 - ab + b^2$$

a perfect square. We can do this by taking a and b to be the legs of an Eisenstein triple.

Clean Cubics

We (all of us) want cubic polynomials

$$f(x) = ax^3 + bx^2 + cx + d$$

with integer coefficients, zeros, extrema, and inflection points.

With a little more work, Eisenstein integers can be used here, too. The problem comes down to finding an Eisenstein integer $\alpha + \beta\omega$ so that

$$N(\alpha + \beta\omega) = 3q^2.$$

For details, see the handout or "Meta-Problems in Mathematics," *College Mathematics Journal*, November, 2000.

$s \rightarrow$ $r \downarrow$	1	2	3
2	$54 - 27x + x^3$	$-128 - 48x + x^3$	
3	$286 - 147x + x^3$	$286 - 147x + x^3$	$-1458 - 243x + x^3$
4	$-506 - 507x + x^3$	$3456 - 432x + x^3$	$-506 - 507x + x^3$
5	$-7722 - 1323x + x^3$	$10582 - 1083x + x^3$	$10582 - 1083x + x^3$
6	$-35282 - 2883x + x^3$	$18304 - 2352x + x^3$	$39366 - 2187x + x^3$

$g(x) = x^3 - 3q^2x + d$ where

$$(1 + 2\omega)(r + s\omega)^2 = m + n\omega$$

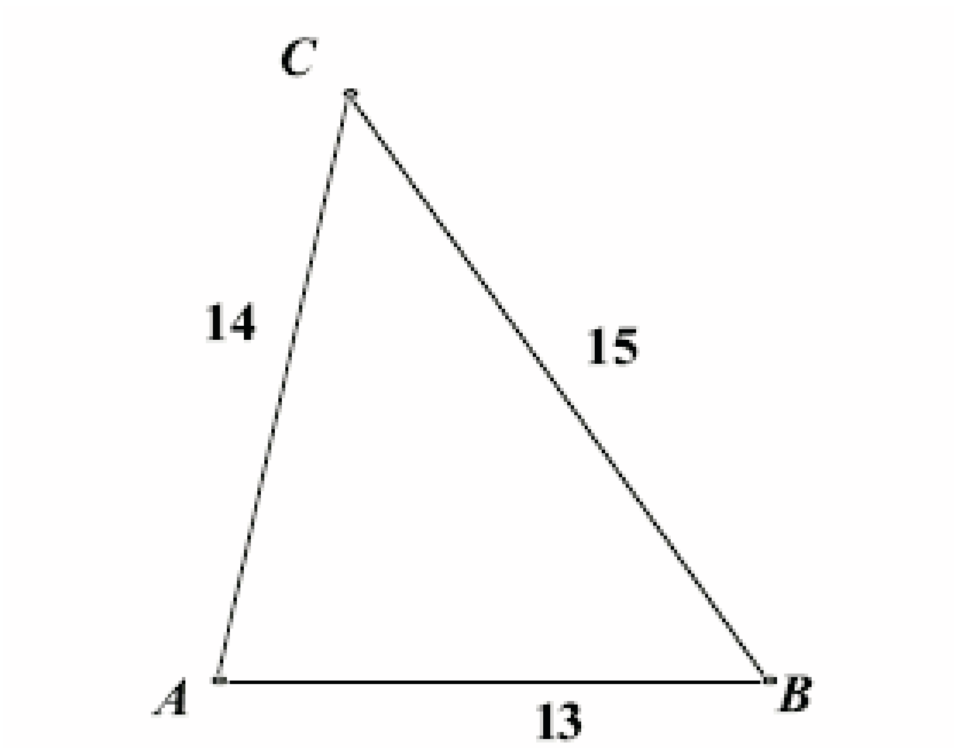
$$3q^2 = \mathrm{N}\left(m + n\omega\right)$$

$$d = m\left(m^2 - 3q^2\right)$$

These can be translated to produce examples with a non-zero x^2 term.

Other Examples

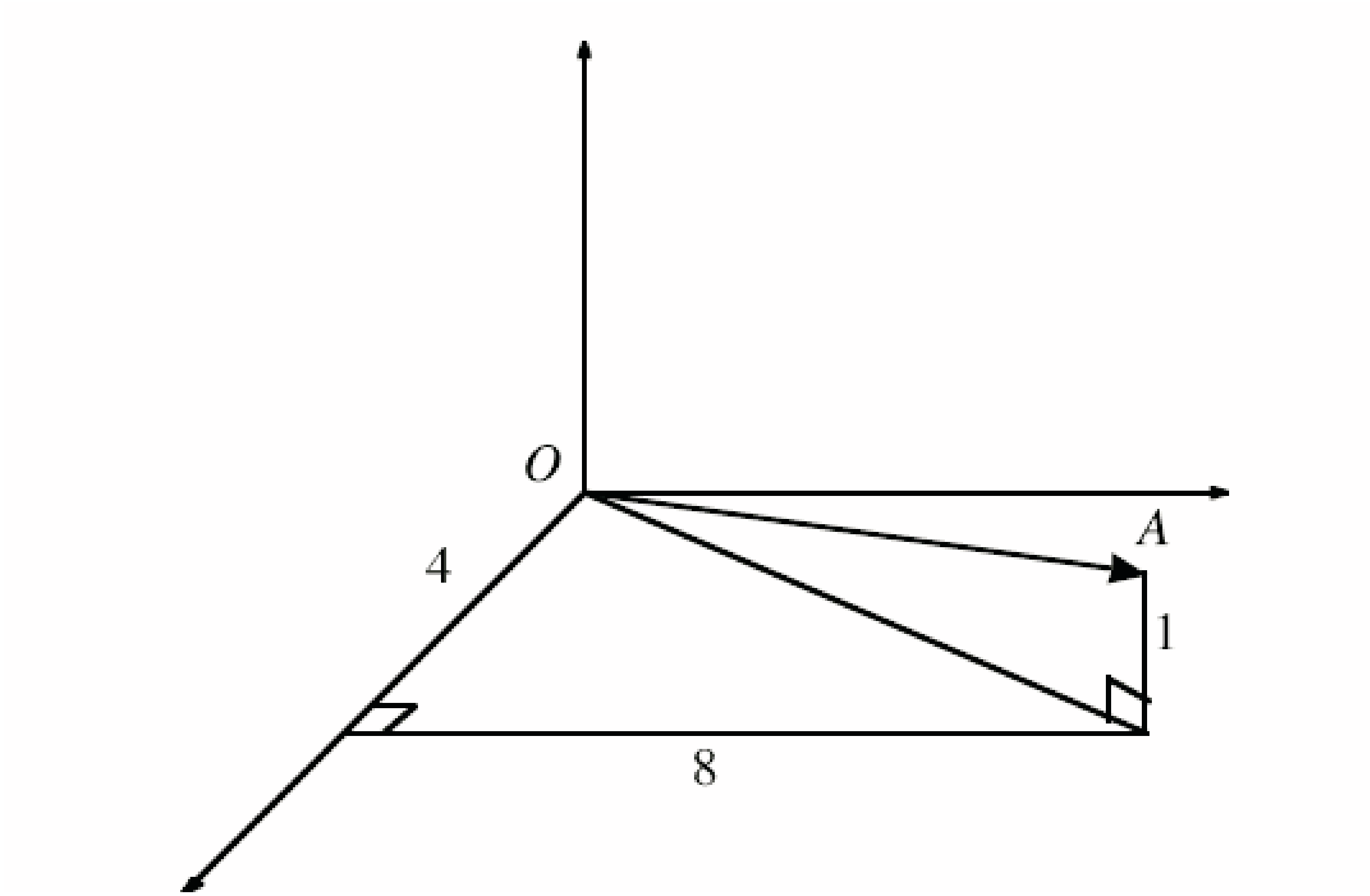
Find *Heron triangles*: Triangles with integral side lengths and area.



A $(13, 14, 15)$ triangle

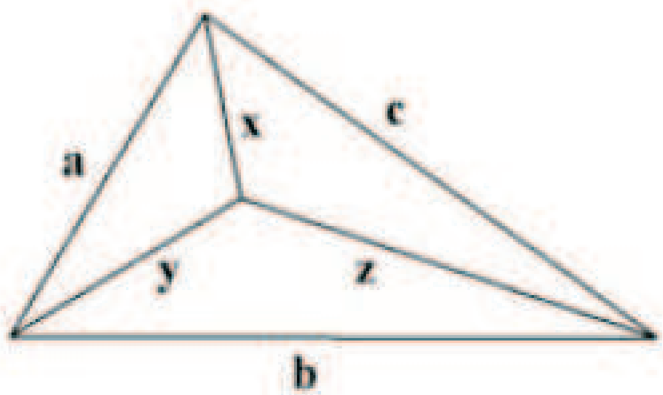
$$
\begin{aligned}
s &= \tfrac{1}{2}(13 + 14 + 15) = 21 \\
A &= \sqrt{21(21-13)(21-14)(21-15)} = 84
\end{aligned}
$$

Find vectors in $\mathbf{Z}^3$ with integral length.
Ex: $(4, 8, 1)$



How long is $\overrightarrow{OA}$?

Find *Matsuura Triples*: integer sided triangles so that the Fermat point is an integer distance from each vertex:



x	y	z	a	b	c
195	264	325	399	511	455
264	325	440	511	665	616
390	528	650	798	1022	910
528	650	880	1022	1330	1232
585	792	975	1197	1533	1365

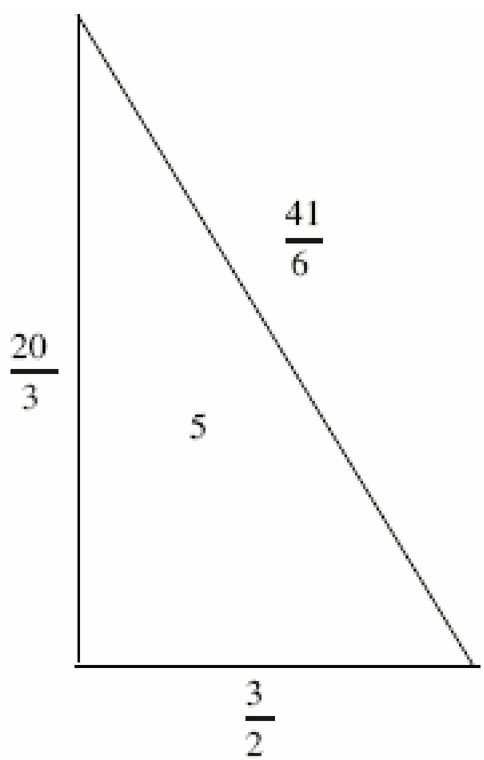
Nice Fermat Triangles

Find *congruent numbers*: Integers that are areas of right triangles with rational side lengths.

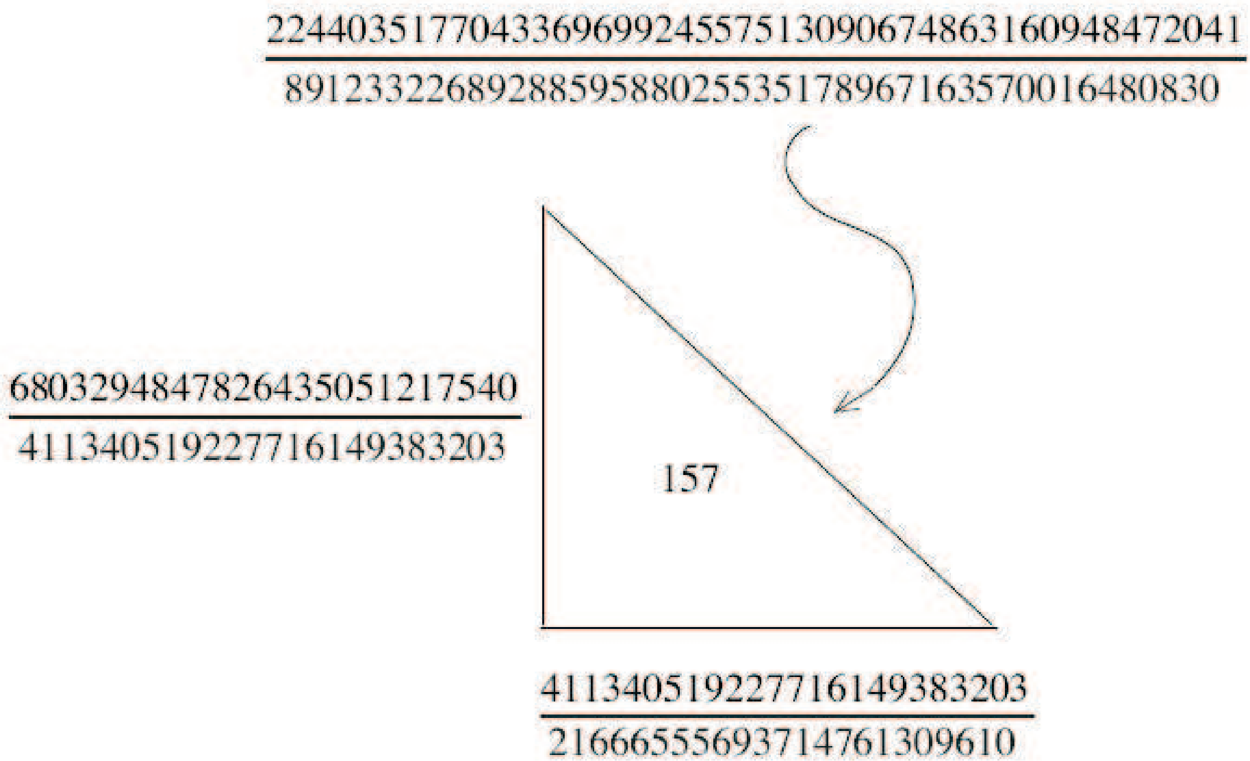
Example: 6 is the area of a $(3, 4, 5)$ right triangle.

Non-example: 1 is not a congruent number (Fermat).

Example: 5 is a congruent number.



Example: 157 is a congruent number .



157 is a congruent number.

Conclusion

There are other examples like this, where mathematics is used in profession-specific ways by teachers.

And a close examination of the teaching profession suggests that useful mathematics for teaching involves a specialized knowledge of the discipline that may not be completely addressed in traditional mathematics preservice courses or professional development programs.

References:

[1] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer Verlag, New York, 1993

[2] A. Cuoco, "Meta-Problems in Mathematics," *College Mathematics Journal*, November, 2000.