ACTIVITY

1

Algebra with Points

Adding and Scaling Points

1. Suppose $A = (3, 1)$ and $B = (-2, 4)$. Calculate each result and plot your answers:

(a) $A + B$ (b) $A + 2B$ (c) $A + 3B$

(d) $A - B$ (e) $A + \frac{1}{2}B$ (f) $A + 7B$

(g) $A - \frac{1}{3}B$ (h) $A + \frac{5}{2}B$ (i) $A - 4B$

2. Suppose $A = (5, -2)$ and $B = (2, 5)$. Calculate each result and plot your answers:

(a) $A + B$ (b) $A + 2B$ (c) $2A + 3B$

(d) $2A - 3B$ (e) $\frac{1}{2}A + \frac{1}{2}B$ (f) $\frac{1}{3}A + \frac{2}{3}B$

(g) $\frac{1}{10}A + \frac{9}{10}B$ (h) $-3A - 4B$ (i) $A - 4B$

As much fun as these calculations are, they are here for more than fun. Try to get a feel for how "linear combinations" work.

3. In each problem, draw a diagram and estimate the coefficients a and b so that $C = aA + bB$. Then find exact values of a and b using algebra.

(a) $A = (3, 0), B = (0, -5), C = (-1, 7)$
(b) $A = (1, 4), B = (4, -1), C = (6, 7)$

4. Suppose that $A = (-2, 4, 1)$, $B = (-3, 0, 3)$, and $C = (5, 6, 2)$. Find

(a) $A+B+C$ (b) $A+B-C$ $2A+\frac{1}{3}B$

(c) $\frac{1}{2}(B+C)$ (d) $\frac{1}{3}A+\frac{2}{3}B$ (e) $\frac{1}{4}A+\frac{3}{4}B$

5. Find A if

(a) $2A+(5,-2,4)=(11,0,4)$
(b) $4A-(3,-3)=A+(6,-9)$

6. Find x, y, and z if $2(x,y,z)+(5,0,3)=(7,4,1)$

7. Find A and B if

$$\begin{cases} 2A+B &= (-1,4,7) \\ A-B &= (-5,1,2) \end{cases}$$

8. Find scalars c_1 and c_2 so that

$$c_1(1,2,4)+c_2(3,2,1)=(-3,2,10)$$

Linear Combinations and Related Stuff

If $\{A_1,A_2,\dots,A_m\}$ are points in $\mathbb{R}^n$, we (all of us) say that a point B is a linear combination of the A_i if you can find (well, if there exist) numbers $c_1,c_2,\dots,c_m$ so that

$$c_1A_1+c_2A_2+\cdots+c_mA_m=B$$

Shorthand for this sum:

$$\sum_{k=1}^{m} c_kA_k = B$$

For example, $(5,7,9)$ is a linear combination of

$$\{(1,2,3),(4,5,6),(7,8,9)\}$$

Because, for example,

$$2(1,2,3)-(4,5,6)+(7,8,9)=(5,7,9)$$

There's another way to do it:

$$(1,2,3)+(4,5,6)+0(7,8,9)$$
$$=(5,7,9)$$

Sometimes this happens.

But $(5,7,10)$ is not a linear combination of

$$\{(1,2,3),(4,5,6),(7,8,9)\}$$

(Why?)

9. Show that $(3,-2,3)$ is a linear combination of $(1,4,1)$ and $(0,2,0)$, but $(3,-3,1)$ is not a linear combination of these.

10. Let $A=(1,3,2)$, $B=(2,0,1)$, and $C=(3,3,2)$.

(a) Write $(1,3,3)$ as a linear combination of A, B, and C.
(b) Write $(1,-3,1)$ as a linear combination of A, B, and C.

11. Let $A=(1,3)$ and $B=(4,-3)$

(a) Write $(-13,36)$ as a linear combination of A and B.
(b) Show that *every* point C in $\mathbb{R}^2$ is a linear combination of A and B.

12. Describe geometrically the set of all linear combinations of

(a) $A=(5,1)$ and $B=(5,5)$, in $\mathbb{R}^2$.
(b) $A=(5,1)$ and $B=(25,5)$, in $\mathbb{R}^2$.
(c) $A=(5,1,3)$ and $B=(5,5,2)$, in $\mathbb{R}^3$.
(d) $A=(5,1,3)$, $B=(5,5,2)$, and $C=(10,6,4)$, in $\mathbb{R}^3$.
(e) $A=(5,1,3)$, $B=(5,5,2)$, and $C=(10,6,5)$, in $\mathbb{R}^3$.

The "set of all linear combinations" is often called "the linear span" by people in the know.

13. Describe geometrically the graphs of the following equations:

(a) $X=(2,3)+t(5,-1),\ t\in\mathbb{R}$.
(b) $X=(1-t)(2,3)+t(5,-1),\ 0\le t\le 1$.
(c) $X=(2,3,-1)+t(5,-1,0),\ t\in\mathbb{R}$.
(d) $X=(1-t)(2,3,-1)+t(5,-1,0),\ 0\le t\le 1$.

Try it with numbers! In each equation, pick some numbers for t, find the corresponding values for X, and plot the result (well, imagine the plot, if you are working in in $\mathbb{R}^3$).

14. In $\mathbb{R}^3$,

(a) What's the equation of the x-y plane?
(b) What's the equation of the y-z plane?
(c) What's the equation of the plane that's parallel to the y-z plane and that passes through $(3,4,5)$?

Vectors, Extensions, and Such

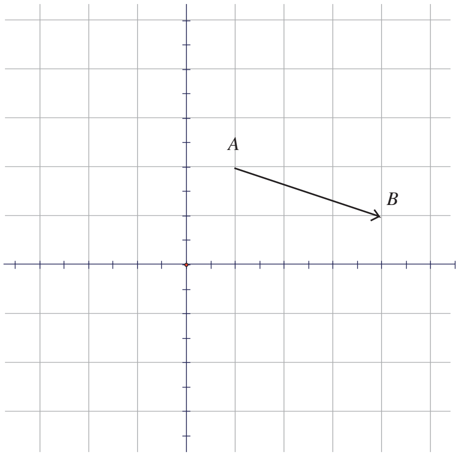
There are many ways to think about vectors. Physicists talk about quantities that have a *magnitude* and *direction* (like velocity, as opposed to speed). Sailors and football coaches draw arrows. Some people talk about "directed" line segments. Mathematics, as usual, makes all this fuzzy talk precise. For us, a vector is nothing other than an *ordered pair of points*.

The gain in precision is accompanied by a loss of all these romantic images carried by words like "magnitude" and "direction."

Definition

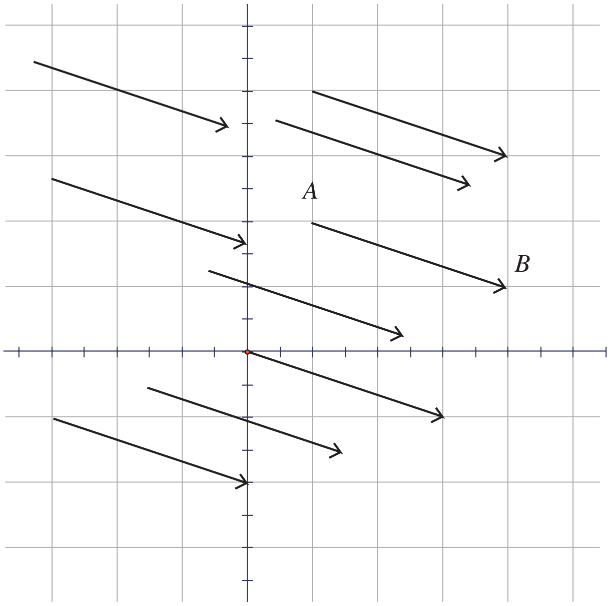
Suppose A and B are points in $\mathbb{R}^n$. The vector from A to B *is the ordered pair of points (A,B). Instead of writing (A,B), we'll write $\overrightarrow{AB}$, but don't be fooled: $\overrightarrow{AB}$ is just an alias for the* *ordered pair* (A, B). *A is called the* tail *of* $\overrightarrow{AB}$ *and B is called the* head *of* $\overrightarrow{AB}$.

In $\mathbb{R}^{2 \text{ or } 3}$, we can picture a vector by a little arrow. So, $\overrightarrow{(1,2)(4,1)}$ can be pictured like this:



Some people like to talk about *equivalent* vectors: vectors with the same magnitude and direction. So, they picture "vector bundles" or "equivalence classes" of vectors. Here are some vectors in the class of $\overrightarrow{(1,2)(4,1)}$:

For example, 50MPH NE starting from Boston is somehow "the same" as 50MPH NE starting from Florence. Of course, it's different in many ways, too.



Which vector in the class starts ay O?

These next three problems help build a "point algebra" description of equivalence, starting from the intuitive notion (same length and direction) in $\mathbb{R}^2$. This will allow us to extend the definition to higher dimensions.

15. Find a vector equivalent to $\overrightarrow{(1,2)(4,1)}$ that starts at

(a) $(-3,4)$
(b) $(3,-4)$
(c) $(0,0)$
(d) (r,s)

16. Prove the "head minus tail" criterion in $\mathbb{R}^2$:

Let $A = (a_1, a_2)$, ... and feel free to use what you know about slope, etc.

Theorem 1.1
If A, B, C, and D are in $\mathbb{R}^2$, $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ if and only if

$$B - A = D - C$$

17. Devise a test for vectors in $\mathbb{R}^2$ to determine whether or not

(a) they are parallel in the same direction
(b) they are parallel in opposite directions

Ways to Think about It

So, we start with this geometric notion of equivalent vectors in the plane—"same length, same direction". In problem 16, we found a way to test for equivalence in the plane that depends only on point algebra—two vectors are equivalent if

$$\text{head minus tail} = \text{head minus tail}$$

It turns out (using some intricate analytic geometry) that this test works in $\mathbb{R}^3$, too. What about in higher dimensions, where we don't have pictures? The tradition is to use the vectorial test (Theorem 1.1 in our case) as the *definition* of equivalence in $\mathbb{R}^n$.

A test that "depends only on point algebra" is sometimes called a *vectorial* test.

This would make a nice way to spend an afternoon. Prove, using coordinates in $\mathbb{R}^3$, that $\overrightarrow{AB}$ has the same length and direction as $\overrightarrow{CD}$ iff $B - A = D - C$.

So, if $A = (-1,2,1,5)$, $B = (-2,3,2,1)$, $C = (0,1,3,0)$, and $D = (-1,2,4,-4)$, we can say that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$. Why? Not for any geometric reason, but because

$$\begin{aligned} B - A &= (-2,3,2,1) - (-1,2,1,5) = (-1,1,1,-4) \quad \text{and} \\ D - C &= (-1,2,4,-4) - (0,1,3,0) = (-1,1,1,-4) \end{aligned}$$

This is a very common program in linear algebra for extending geometric vocabulary to higher dimensions. It goes like this:

The Extension program for definitions

1. Take a familiar geometric idea in 2 and 3 dimensions.
2. Find a way to describe that idea using only the algebra of points.
3. Use that vectorial description as the *definition* of the idea in dimensions higher than 3.
4. Prove that this extended definition is consistent with all your other definitions.

Many people say that they can "see" things in 4 or 5 dimensions. Be skeptical. You can certainly see calculations with points in $\mathbb{R}^{4 \text{ or } 5}$, and you can illustrate what you see with drawings in $\mathbb{R}^{2 \text{ or } 3}$, and, once internalized, this all becomes a coherent package in your mind, so that it feels like you are experiencing higher dimensions. So, maybe they are right . . . .

As we'll see, part 4 above is the hard part of this program. For example, when we extend the notion of length to higher dimensions (what would be a good way to do this?), we should prove that equivalent vectors in dimensions higher than 3 do, in fact, have the same "length."

So, let's use the extension program right away and define equivalent and parallel vectors:

Definition

Suppose A, B, C, and D ate points in $\mathbb{R}^n$.

- *We say that $\overrightarrow{AB}$ is* equivalent *to $\overrightarrow{CD}$ if*

$$B - A = D - C$$

- *We say that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ if there is a number r so that*

$$B - A = r(D - C)$$

  If $r > 0$, we say that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction. *If $r < 0$, we say that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the* opposite direction.

This naked definition is often just dropped into a linear algebra book with no justification. This causes heartburn for many students.

In $\mathbb{R}^2$, it's clear that every vector is equivalent to a vector whose tail is the origin (right?). Using our fancy new way to think of equivalence, we can show that the same thing happens in $\mathbb{R}^n$:

Theorem 1.2

If A, B are points in $\mathbb{R}^n$, $\overrightarrow{AB}$ is equivalent to a vector starting at the origin. In fact, $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O\,(B-A)}$.

Proof. Don't think "same length, same direction." Think "head minus tail = head minus tail". Then it's almost too simple.

We don't even know what length means in $\mathbb{R}^n$. What would a good definition?

Since $B - A = (B - A) - O$, the "head minus tail" on $\overrightarrow{AB}$ is the same as the "head minus tail" on $\overrightarrow{O\,(B-A)}$, so the two vectors are equivalent by definition.

Facts and Notation

Theorem 1.2 tells us that every vector "bundle" contains a representative that starts at the origin. We'll use this to adopt a convention. From now on, when we see a point A in $\mathbb{R}^n$, we'll think about it in one of three ways:

1. as a point in $\mathbb{R}^n$,
2. as a vector in $\mathbb{R}^n$ that starts at the origin: $\overrightarrow{OA}$,
3. as a representative of the class of vectors in $\mathbb{R}^n$ that are all equivalent to $\overrightarrow{OA}$.

How do we know which mental image to use? As English teachers like to say, the context will make it clear.

Actually, sometimes it won't be clear from the context—it will be useful to think of the same A in more than one way at the same time. You get used to this.

Time for some problems.

18. Determine which pairs of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are

1. equivalent
2. parallel in the same direction
3. parallel in opposite directions

As usual, this is not busy work. there are theorems buried here. To help find them, plot the points whenever you are in $\mathbb{R}^{2 \text{ or } 3}$.

(a) $A = (3, 1), B = (4, 2), C = (-1, 4), D = (0, 5)$

(b) $A = (3, 1), B = (4, 2), C = (0, 5), D = (-1, 4)$

(c)

$$A = (3, 1, 5), B = (-4, 1, 3), C = (0, 1, 0), D = (14, 1, 4)$$

(d)

$$A = (-4, 1, 3), B = (3, 1, 5), C = (0, 1, 0), D = (14, 1, 4)$$

(e)

$$A = (-1, 2, 1, 5), B = (0, 1, 3, 0), C = (-2, 3, 2, 1), D = (-1, 2, 4, -4)$$

(f)

$$A = (-1, 2, 1, 5), B = (-2, 3, 2, 1), C = (0, 1, 3, 0), D = (-1, 2, 4, -4)$$

(g) $A = (1, 3), B = (4, 1), C = (-2, 3), D = (13, -7)$
(h) $A = (3, 4), B = (5, 6), C = B - A, D = O$
(i) $A = 0, B = (4, 7), C = (5, 2), D = B + C$

19. Find a point P if $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$, where

$$A = (2, -1, 4), B = (3, 2, 1), \text{ and } Q = (1, -1, 6)$$

20. Suppose

$$A = (3, 1, -1, 4), B = (1, 3, 2, 0), C = (1, 1, -1, 3) \text{ and } D = (-3, a, b, c)$$

Find a, b, and c if $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

21. **PODASIP**. If $\overrightarrow{AB}$ is equivalent to $\overrightarrow{AC}$, then $B = C$. True in $\mathbb{R}^n$.

22. In $\mathbb{R}^2$, show that if A, B, and O are collinear, then $B = cA$ for some number c.

23. In $\mathbb{R}^n$, suppose $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$. Show that $\overrightarrow{AC}$ is equivalent to $\overrightarrow{BD}$. Illustrate geometrically in $\mathbb{R}^2$.

24. In $\mathbb{R}^2$, let ℓ be the line with equation $5x + 4y + 20 = 0$. If $P = (-4, 0)$ and $A = (4, -5)$, show that ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A.

Context clues: In "$\overrightarrow{PQ}$ is parallel to A," P and Q are points, but A is a vector. What vector?

25. Let $P = (3, 0)$ and $A = (1, 5)$. If ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A, find the equation of ℓ.

26. In $\mathbb{R}^3$, let $A = (1, 0, 2)$, $B = (1, 0, 3)$. Find the equation of the plane containing A and B.

Plane containing A and B? Don't you need three points? Not if you think of A and B as *vectors*.

27. Show that the following definition of midpoint in $\mathbb{R}^n$ really does give you the midpoint in $\mathbb{R}^2$:

Definition

If A and B are points in $\mathbb{R}^n$, the midpoint of $\overrightarrow{AB}$ is

$$\frac{1}{2}(A + B)$$

28. In $\mathbb{R}^4$, let $A = (-3, 1, 2, 4)$, $B = (5, 3, 6, -2)$, and $C = (1, 2, -2, 0)$. Using the definition of midpoint from problem 27, suppose that M is the midpoint of $\overrightarrow{AB}$ and N is the midpoint of $\overrightarrow{AC}$. Show that $\overrightarrow{MN}$ is parallel to $\overrightarrow{BC}$.

What fact from plane geometry does this generalize? Can you prove it for any points A, B, and C? Try it

ACTIVITY

2 Dot Product, Length, and Angle

The object of the game is to extend notions of angle and distance to $\mathbb{R}^n$. According to our "extension program," we should find a way to characterize these things with vectors in the situations we can actually see.

If we were in England (or Sturbridge Village), we'd call it the "extension programme."

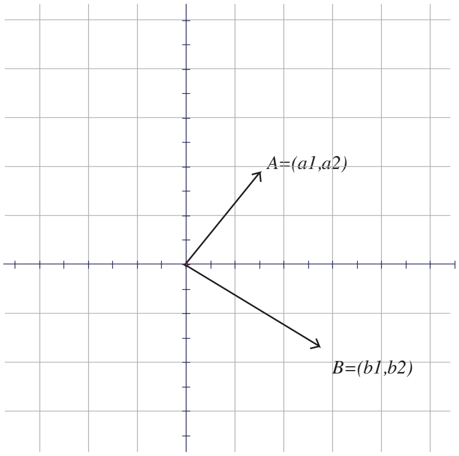
Perpendicularity in two and three dimensions

Let's start in $\mathbb{R}^2$ and try to find a vector characterization of perpendicularity. How can we tell if vectors $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are perpendicular?

"Vectors A and B?" Remember, this means "vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$."

1. **For teachers of Algebra 2.** Can someone prove for us that two lines are perpendicular $\Leftrightarrow$ their slopes are negative reciprocals?

The situation looks like this:



So, the "slope of A" is $\frac{a_2}{a_1}$, the slope of B is $\frac{b_2}{b_1}$, and the vectors are perpendicular $\Leftrightarrow$

$$\frac{a_2}{a_1} = -\frac{b_1}{b_2}$$

This is the same as saying

$$a_1b_1 + a_2b_2 = 0$$

It's actually better when written this way. Why?

It turns out that the same thing is true in $\mathbb{R}^3$: If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, 3_3)$, then

$$A \perp B \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$$

Proving this makes for a nice exercise in three-dimensional geometry. Try it sometime.

Try it sitting under a tree with a cognac.

Example

Suppose $A = (1, 3, 7)$. If $B = (4, 1, -1)$, $A \perp B$ because

$$1 \cdot 4 + 3 \cdot 1 + 7 \cdot (-1) = 0$$

Facts and Notation

There's a strange vocabulary convention in all this. We call lines that meet at a right angle "perpendicular," but we call vectors that meet at a right angle (at their common tail) "orthogonal." One is Latin, the other is Greek. Go figure.

Also, using the vectorial characterizations of orthogonality:

$$\begin{aligned} a_1b_1 + a_2b_2 &= 0 \quad \text{in } \mathbb{R}^2\text{, and} \\ a_1b_1 + a_2b_2 + a_3b_3 &= 0 \quad \text{in } \mathbb{R}^3 \end{aligned}$$

We have to allow that the origin is orthogonal to every vector.

So, we have a new way to describe orthogonality in 2 and 3 dimensions: we "multiply" the vectors in question, coordinate by coordinate, add the answers, and check to see if we get 0. This operation of summing the coordinate-wise products can be carried out in $\mathbb{R}^n$, and we call it the dot product.

Definition

Suppose $A = (a_1, a_2, \ldots, a_n)$ *and* $B = (b_1, b_2, \ldots, b_n)$ *are vectors in* $\mathbb{R}^n$*. We define the dot product of* A *and* B*, written* $A \cdot B$ *to be the number calculated via this formula:*

$$A \cdot B = a_1 b_1 + a + 2b_2 + \cdots + a_n b_n$$

$$= \sum_{i=1}^{n} a_i b_i$$

For example, $(1, 3, 2, 4) \cdot (0, 1, -1, 2) = 1 \cdot 0 + 3 \cdot 1 + 2 \cdot -1 + 4 \cdot 2 = 9$.

Armed with our new definition, we can describe orthogonality in 2 and 3 dimensions more succinctly:

Theorem 1.3

If A *and* B *are two vectors in* $\mathbb{R}^{2\ or\ 3}$,

$$A \perp B \Leftrightarrow A \cdot B = 0$$

Feel a definition coming on? We apply our extension program to define orthogonality in higher dimensions via the vectorial descriptions in places we can see.

Definition

If A *and* B *are vectors in* $\mathbb{R}^n$*, we say that* A *is* orthogonal *to* B*, and we write* $A \perp B$*, if* $A \cdot B = 0$*.*

2. (a) Find four vectors orthogonal to $(3, 4)$.
  (b) What is the set of all vectors orthogonal to $(3, 4)$? Find an equation for this set.

3. What is the set of all vectors orthogonal to *both* $(3, 4)$ and $(5, 1)$? Find an equation for this set.

4. (a) Find four vectors orthogonal to $(3, 4, -1)$.
  (b) What is the set of all vectors orthogonal to $(3, 4, -1)$? Find an equation for this set.

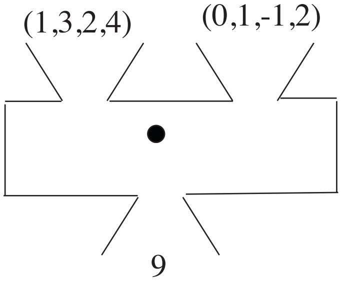
5. What is the set of all vectors orthogonal to *both* $(3, 4, -1)$ and $(6, 1, -1)$? Find an equation for this set.

6. What is the set of all vectors orthogonal to *all of* $(3, 4, -1)$, $(1, 3, 4)$, and $(6, 1, -1)$? Find an equation for this set.

7. Suppose $A = (3, 4)$, $B = (9, 8)$, and $C = (6, -5)$. One angle of $\triangle ABC$ is a right angle. Which one is it?

8. Suppose $A = (5, 3, 3)$, $B = (1, 3, 1)$, and $C = (2, 6, -1)$. One angle of $\triangle ABC$ is a right angle. Which one is it?

So, we have a new operation, one that takes two *vectors* (or points, if you want) and produces a *number*. Right now, it's nothing other than that—a new operation in the algebra of points.



Before we start assigning it properties, let's play with it a bit, seeing how it is a convenient shorthand for things with which we are already familiar and investigating how it interacts with other operations. (adding and scaling, for example)

"Non-zero" means "not equal to O." You can think of A and B as points (instead of vectors) if that makes you happy.

9. Suppose A and B are non-zero vectors in $\mathbb{R}^n$, and c is a number. Characterize each of these expressions as "vector," "number," or "meaningless."

(a) $A \cdot (cB)$ (b) $(A \cdot B)A$ (c) $(A \cdot B) + A$

(d) $(A \cdot A)B + (B \cdot B)A$ (e) $c^2(A \cdot A)$ (f) $A - \dfrac{A \cdot B}{B \cdot B}B$

10. Suppose that $A = (1, 3, 2)$ and $B = (4, 1, -4)$. What is the value of the following expression?

$$B \cdot \left(A - \frac{A \cdot B}{B \cdot B}B\right)$$

11. Let $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n)$, and $C = (c_1, c_2, \ldots, c_n)$ be vectors in $\mathbb{R}^n$ and let s be a number. Show that the following statements are true in $\mathbb{R}^n$.

(a) $A \cdot B = B \cdot A$
(b) $A \cdot sB = sA \cdot B = s\,(A \cdot B)$

(c) $A \cdot (B + C) = A \cdot B + A \cdot C$
(d) $A \cdot A \geq 0$ and $A \cdot A = 0 \Leftrightarrow A = O$.

12. Suppose A and B are vectors, $\|A\| = a$, and $\|B\| = b$. Show that

$$\|bA\| = \|aB\|$$

Length

13. Show that the following definition for length makes sense in 2 and 3 dimensions:

Definition
If $A \in \mathbb{R}^n$, the length of A, written $\|A\|$ is defined by the formula

$$\|A\| = \sqrt{A \cdot A}$$

14. Find a vector in the same direction as A=(3,2,6) that has length 1. Generalize.

15. Establish the following properties of length:

(a) $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = O$
(b) $\|cA\||c|\,\|A\|$ for numbers c and vectors A.
(c) $\|A + B\| \leq \|A\| + \|B\|$

The last one is not easy with what we have now. But illustrate it with a picture.

16. Show that the following definition for distance makes sense in 2 and 3 dimensions:

Definition
If $A, B \in \mathbb{R}^n$, the distance from A to B, written $d(A, B)$ is defined by the formula

$$d(A, B) = \|B - A\|$$

17. Suppose $A = (1, -3, 4, 2)$, $B = (4, 1, 16, 2)$, and $C = (6, -3, 4, 14)$. Show that $\triangle ABC$ is isosceles.

18. Suppose A and B are vectors (in $\mathbb{R}^2$, say). How long is each of the diagonals of the parallelogram whose vertices are O, A, B, and $A + B$?

19. Suppose A and B are vectors. Show that

(a) $\|A + B\|^2 = \|A\|^2 + \|B\|^2 + 2\, A \cdot B$
(b) $\|A + B\|^2 - \|A - B\|^2 = 4\, A \cdot B$
(c) $\|A + B\|^2 + \|A - B\|^2 = 2\|A\|^2 + 2\|B\|^2$

Interpret part 19c geometrically.

20. SuposeA $\perp$ B. Show that

$$\|A+B\|^2 = \|A\|^2 + \|B\|^2$$

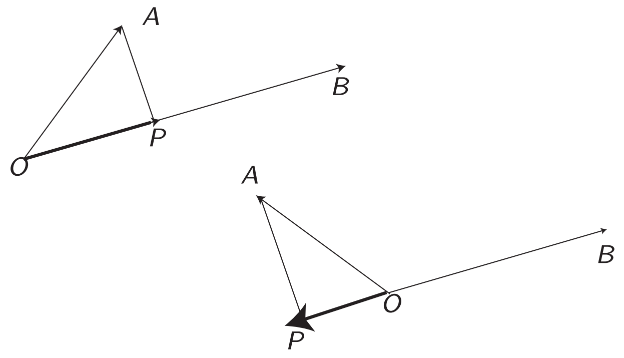
Is the converse true?

What famous theorem from geometry is this?

Definition

Suppose A and B are vectors. By the projection of A along B, *we mean the vector P so that*

1. *P is in the direction of B (or in the opposite direction), and*
2. $\overrightarrow{PA} \perp B$



21. Suppose $A = (5,2)$, $A' = (-2,5)$, and and $B = (6,0)$. Find the projection of A along B and the projection of A' along B.

22. Suppose $A = (3,1)$, $A' = (2,4)$, and and $B = (6,2)$. Find the projection of A along B and the projection of A' along B.

23. Suppose $A = (2,0,1)$ and $B = (6,4,2)$. Find the projection of A along B.

24. Suppose A and B are vectors. Derive a formula for the projection of A along B.

We denote the the projection of A along B as $\text{Proj}_B A$.

25. If $A = (-3,1,-2,4)$ and $B = (1,1,2,0)$, find $\text{Proj}_B A$ and $\text{Proj}_A B$

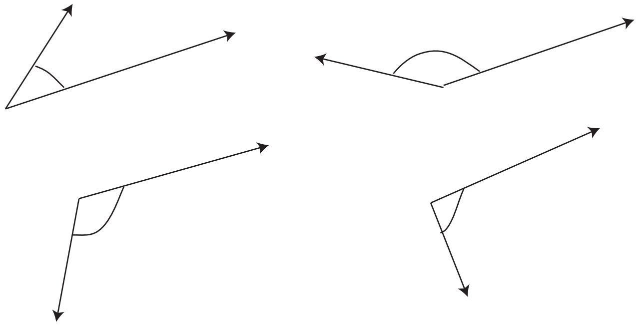
26. If A and B are vectors, show that

$$\|\text{Proj}_B A\| = \frac{|A \cdot B|}{\|B\|}$$

Angle

Every pair of vectors in $\mathbb{R}^{2 \text{ or } 3}$ determines a unique angle between 0 and 180°.

In radians, this angle is between 0 and π.



We'd like to find a formula for this angle in terms of the vectors, and we'd like to extend this formula as a definition to $\mathbb{R}^n$.

27. Find the angle between each pair of vectors:

(a) $(5, 5)$ and $(7, 0)$
(b) $(-5, 5)$ and $(7, 0)$
(c) $(-5, 5)$ and $(6, 6)$
(d) $(-5, 5)$ and $(6, 6)$
(e) $(\sqrt{3}, 1)$ and $(0, 6)$
(f) $(-3, 4)$ and $(5, 12)$

Some Trigonometry

We'll return to trigonometry later, but for now we need to use the "unit circle" definition of cosine. For this, it might help to look at a Sketchpad demo.

The point we need is that, for angles between 0 and π, knowing the cosine is as good as knowing the angle.

"Cosine is 1-1 on $(0, \pi)$."

28. Suppose A and B are vectors in $\mathbb{R}^2$ that make an acute angle θ. Show that

This result is true even if θ is not acute,

$$\cos\theta = \frac{A \cdot B}{\|A\| \, \|B\|}$$

For Discussion

What details would we have to nail down if we wanted ot make the following definition:

And then let's nail them down.

Definition

If A and B are vectors in $\mathbb{R}^n$, the angle between A and B is that unique angle θ between 0 and π so that

$$\cos\theta = \frac{A \cdot B}{\|A\|\,\|B\|}$$

---

29. Find thecosine of the angle between each pair of vectors.

(a) $A = (3,4)$, $B = (0,7)$

(b) $A = (1,1,1)$, $B = (1,1,0)$

(c) $A = (2,1,0)$, $B = (5,-3,4)$

(d) $A = (-3,1,2,5)$, $B = (4,1,3,-4)$

30. (a) Show that the angle between $A = (1,1)$ and $B = (1,\sqrt{3})$ is $\frac{\pi}{12}$ (or $15°$).

(b) Find an exact value of $\cos\frac{\pi}{12}$.

31. If A and B are vectors that make an angle of θ, show that

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\,\|A\|\,\|B\|\cos\theta$$

What famous theorem is this?

32. Suppose $A = (\sqrt{3},\sqrt{3},1)$, $B = (-1+\sqrt{3}, 1+\sqrt{3}, 1)$, and $C = (-1,1,1)$. Show that $\triangle ABC$ is a 30-60-90 triangle and verify that the angle opposite the smallest angle is half as long as the hypoteneuse.

ACTIVITY

3 *Matrices*

Matrices can be used to move points around. Here's how:

Why do we write the points as a column? Tradition.

$$\begin{pmatrix} 3 & 7 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 3\cdot 5 + 7\cdot(-4) \\ 2\cdot 5 + 1\cdot(-4) \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \end{pmatrix}$$

So, we say "The matrix $\begin{pmatrix} 3 & 7 \\ 2 & 1 \end{pmatrix}$ sends $(5,-4)$ to $(-13,6)$", or "the image of $(5,-4)$ under $\begin{pmatrix} 3 & 7 \\ 2 & 1 \end{pmatrix}$ is $(-13,6)$." In each of these problems, draw $\triangle ABC$ and then move each of the vertices via the matrix M. Describe what happens geometrically.

1. $A=(4,3)$, $B=(7,4)$, $C=(5,5)$; $M=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. $A=(4,3)$, $B=(7,4)$, $C=(5,5)$; $M=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3. $A=(4,3)$, $B=(7,4)$, $C=(5,5)$; $M=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

4. $A=(-4,3)$, $B=(7,4)$, $C=(1,5)$; $M=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

5. $A=(0,0)$, $B=(3,1)$, $C=(1,-1)$; $M=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

6. $A=(0,1)$, $B=(3,1)$, $C=(1,-1)$; $M=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

7. $A = (0,1)$, $B = (3,1)$, $C = (1,-1)$; $M = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

8. $A = (0,1)$, $B = (3,1)$, $C = (1,-1)$; $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

9. $A = (0,1)$, $B = (3,1)$, $C = (1,-1)$; $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

10. $A = (4,3)$, $B = (7,4)$, $C = (5,5)$; $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Notation. Suppose M is a matrix, A is a point, and n is a positive integer. The expression $M^n A$ means "apply M to A, apply M to the resulting point, and keep doing this n times."

For example, suppose $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $A = (0,1)$. Then $M^3 A$ means

Oh my. Notice the points: $(0,1), (1,2), 2,3)(3,5)$.

$$
\begin{aligned}
M\big(M\,(MA)\big) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \underbrace{\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}_{\downarrow} \right] \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \underbrace{\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]}_{\downarrow} \\
&= \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\downarrow} \\
&= \begin{pmatrix} 3 \\ 5 \end{pmatrix}
\end{aligned}
$$

So, after three applications of M, $(1,0)$ ends up at $(3,5)$. This process is called *iterating* M three times on (0,1).

11. Suppose $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find

(a) $M^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (b) $M^5 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (c) $M^6 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

12. Suppose $M = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}$.

(a) Plot the points $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

A calculator or computer might help here.

(b) Pick another point $A \neq (1, 1)$ and plot the points $M^n A$ for $n = 1, \ldots, 10$ on the same set of axes.

Write A as a column, like $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) Try it for some other "seeds" A—pick some way up and close to the y-axis, some way out and close to the x-axis, some in the middle of each quadrant. Have fun.

(d) Find some points $A = (a, b)$ so that

$$MA = \begin{pmatrix} ca \\ cb \end{pmatrix}$$

for some number c.

13. Suppose $M = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) Plot the points $M^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

(b) Pick another point $A \neq (2, 1)$ and plot the points $M^n A$ for $n = 1, \ldots, 10$ on the same set of axes.

(c) Try it for some other "seeds" A—pick some way up and close to the y-axis, some way out and close to the x-axis, some in the middle of each quadrant.

(d) Find some points $A = (a, b)$ so that

$$MA = \begin{pmatrix} ca \\ cb \end{pmatrix}$$

for some number c.

14. Who directed *The Matrix*?

Just kidding.

15. Suppose $M = \begin{pmatrix} 41 & -12 \\ -12 & 34 \end{pmatrix}$.

(a) Plot the points $M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

(b) Pick another point $A \neq (1, 1)$ and plot the points $M^n A$ for $n = 1, \ldots, 10$ on the same set of axes.

(c) Try it for some other "seeds" A—pick some way up and close to the y-axis, some way out and close to the x-axis, some in the middle of each quadrant.

(d) Find some points $A = (a, b)$ so that

$$MA = \begin{pmatrix} ca \\ cb \end{pmatrix}$$

for some number c.

16. Suppose $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Plot the points $M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

17. Suppose

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \frac{109}{30} \end{pmatrix}$$

(a) Plot the points $A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

(b) Plot the points $A^n \begin{pmatrix} .3 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

(c) Plot the points $A^n \begin{pmatrix} \frac{10}{3} \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

18. Let

$$B = \begin{pmatrix} \frac{21}{10} & -\frac{6}{5} \\ -\frac{6}{5} & \frac{7}{5} \end{pmatrix}$$

(a) Find all points D so that BD, D and the origin, $(0, 0)$, are collinear.

(b) Plot the points $B^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $n = 1, \ldots, 10$ on the same set of axes.

If you want to apply a matrix to several points at once, you can use the following scheme: Suppose $A = (4, 3)$, $B = (7, 4)$,

$C = (5,5)$, and $M = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Instead of calculating like this:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix}$$

We can do it all at once and write:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 7 & 5 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 5 \\ 11 & 18 & 15 \end{pmatrix}$$

And you can forget that the second matrix was a bunch of points and just think of it as another matrix. In this way, you can "multiply" the matrices $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 4 & 7 & 5 \\ 3 & 4 & 5 \end{pmatrix}$ to get the matrix $\begin{pmatrix} 4 & 7 & 5 \\ 11 & 18 & 15 \end{pmatrix}$.

19. Suppose $M = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix}$. Find

(a) MN (b) NM

(c) $(MN)M$ (d) $M(NM)$

(e) M^2 (that is, MM) (f) M^2N

20. Suppose $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find

(a) M^2 (b) M^3 (c) M^4

20. Suppose $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Find M^5. What is M^n (n a positive integer)?

21. What would be a good way to add matrices?

22. Find a matrix I so that $AI = IA = A$ for every 2×2 matrix A.

The matrix I is called the *identity matrix.* Is it unique?

23. Suppose $P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ Find, if possible, a matrix Q so that $PQ = QP = I$.

The matrix Q is called the *multiplicative inverse* of P and is sometimes written P^{-1}. Is it unique?

24. Suppose $P = \begin{pmatrix} 3 & 15 \\ 2 & 10 \end{pmatrix}$ Find, if possible, a matrix P^{-1} so that $PP^{-1} = P^{-1}P = I$.

25. Develop test to tell whether or not a matrix P has a multiplicative inverse.

26. Suppose

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

Find:

(a) $P^{-1}AP$ (b) $(P^{-1}AP)^2$ (c) $(P^{-1}AP)^5$

(d) $P^{-1}AP$ (e) $P^{-1}A^2P$ (f) $P^{-1}A^5P$

(g) A^5 (h) A^6 (i) A^n (n a positive integer)

27. Suppose

$$A = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$$

Find:

(a) $P^{-1}AP$ (b) $(P^{-1}AP)^2$ (c) $(P^{-1}AP)^5$

(d) $P^{-1}AP$ (e) $P^{-1}A^2P$ (f) $P^{-1}A^5P$

(g) A^5 (h) A^6 (i) A^n (n a positive integer)

28. Let $M = \begin{pmatrix} 0 & 1 \\ -21 & 10 \end{pmatrix}$

(a) Find a matrix P so that$P^{-1}AP$ is a diagonal matrix.
(b) Use part 28a to find a formula for A^n (n a positive integer).

The process in problem 28a is called *diagonalization*.

A diagonal matrix is a matrix of the form

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.$$

29. Suppose

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(a) Find a matrix P so that $P^{-1}AP$ is a diagonal matrix.
(b) Use part 29a to find a formula for A^n (n a positive integer).

If P has a multiplicative inverse and A is any matrix, the matrix $P^{-1}AP$ is said to be *similar* to A.

Extensions and Extras

30. Suppose that $M = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$. The unit square is the square whose vertices are the origin, $(1,0)$, $(1,1)$, and $(0,1)$.

(a) Sketch the quadrilateral whose vertices are the points you get you get if you apply M to the vertices of the unit square.
(b) How does the area of this quadrilateral compare to the area of the unit square.

31. Extend problem 30 by looking at the effect on the unit square of other matrices M. Develop a theory that describes the effect on area when a matrix is applied to the vertices of a polygon.

32. Suppose that $M = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$, and pick two points A and B.

(a) If C is the midpoint of $\overline{AB}$, is MC the midpoint of the segment between MA and MB?

(b) If C is between A and B, is MC between MA and MB?

33.

(a) Pat is baking two types of muffins. His blueberry muffin recipe requires 2 lbs of sugar and 3 lbs of flour per dozen. And his carrot muffin recipe calls for 1 lb of sugar and 4 lbs of flour per dozen. He plans to bake 8 dozen blueberry muffins and 11 dozen carrot muffins. How many pounds of sugar and flour must he buy?

(b) Complete the following multiplication:

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

(c) Compute each of the following:

- $\begin{pmatrix} 3 & 7 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$
- $\begin{pmatrix} -1 & 2 \\ 5 & -7 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$

34. Given a 2 by 2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we call $ad-bc$ its *determinant*. Compute the determinants of the following matrices:

- $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$
- $\begin{pmatrix} 11 & 3 \\ 7 & 2 \end{pmatrix}$
- $\begin{pmatrix} 4 & 2 \\ 9 & 5 \end{pmatrix}$

35. Find the inverse matrix:

- $\begin{pmatrix} 4 & 2 \\ 9 & 5 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 3 & 9 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 2 & 1 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

36. Explain why the matrix

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

is *non-invertible*.

37. Consider the following system of equations

$$\begin{aligned} 2x + 5y &= 11 \\ 3x + 8y &= 17 \end{aligned}$$

(a) Rewrite the system in matrix form.

(b) Solve for (x, y) using matrix multiplication.