Some Ideas for Fourth Year Courses

Al Cuoco

Center for Mathematics Education

EDC

acuoco@edc.org

Slides available at `http://www.edc.org/CME/showcase/`

Example 1: Linear Algebra

Some History

- 1971–1974: NSF Teacher Institute at Bowdoin
- 1972–1993: The course at Woburn High
- 1995–1998: Gateways to Advanced Mathematical Thinking
- 1996–2008: The *CME Project*
- 2008: Linear Algebra and Geometry

Why Linear Algebra in High School?

- There are few prerequisites
  - no limits or ∞
- It is constructive
  - theorems are accompanied by algorithms
- It is parsimonious
  - exploits a very few general purpose tools
- Results and proofs emerge from experience
  - numerical examples $\Rightarrow$ general results
- The mathematics is *important*

The Woburn High Course

Term 1

- Vector geometry and vector algebra
  - Arithmetic with points
  - Vectors
  - Lines and planes: Geometry in $\mathbf{R^n}$
  - Linear (in)dependence, linear span

Term 2

- The solution of linear equations
  - Gauss-Jordan elimination
  - The "fatter than tall" theorem

- Matrices and matrix algebra
  - Adding and scaling: algebraic properties
  - Matrix multiplication
  - Matrices as functions
  - Kernel, image, and rowspace of a matrix
  - Classification of the solution sets for systems $AX = B$
  - Transformations of $\mathbf{R}^{2\text{or}3}$

Term 3

- Vector spaces
  - $\mathbf{R^n}$, matrices, polynomials, $\mathbf{C}$
  - Definition of vector space
  - Subspaces
  - Linear span and generating systems
  - Basis and dimension
  - Coordinate vectors
  - Dimension theorems and invariance of dimension
  - Subspaces associated with a matrix
  - Dimension with coordinate vectors

Term 4

- Linearity and linear mappings
  - Kernel and image
  - Extension by linearity
  - Matrices and linear maps
  - Change of basis and conjugation
- Determinants
  - Determinants as area and volume
  - Cramer's rule
  - Determinants and linear independence
- Eigenvalues and eigenvectors

The Plan for Linear Algebra and Geometry

- **Resources for students**: a three-semester library of student resources, complete with teacher editions, solution manuals, and software enhancements that can be customized to create courses for students and teachers that range in length from one to three semesters.

- **Resources for teachers**: a high school linear algebra community and professional development program that will create the capacity for teaching linear algebra in more (and more diverse) high schools.

Student Resources

Core Semester

- **Algebra with Points and Vector Geometry.**
- **Angles, Lines, and Planes.**
- **The Solution of Linear Systems.**
- **Matrix Algebra.**

Possible Standalone Modules

- Digital Image Processing.
- Computer Graphics.
- Statistics and Linear Algebra.
- Google Page Rank.
- Recurrence Equations.
- Fractals, Ferns, and Iteration.
- Markov Chains.
- Transformations and Quadratic Forms.
- Vector Spaces.
- Linearity and Linear Maps.
- Determinants, Eigenvalues, and Eigenvectors.

Teacher Resources

- Curricular resources
  - Teacher Editions
  - Solution Guides
  - Answer keys
- Professional resources
  - An annual summer conference
    - Overview of the core semester
    - "Perspectives on linear algebra"
  - Online communities

Our uses of technology

- Dynamic geometry environment
- Computer algebra system
- Function modeling language

are used to

- reduce computational overhead
- experiment with mathematical objects
- build models of mathematical phenomena

Software

- TI – Nspire^{tm}

- Matlab

People

EDC

- Al Cuoco
- Paul Goldenberg
- Wayne Harvey
- Bowen Kerins
- Helen Lebowitz
- Kevin Waterman

Core Consultants

- Tom Banchoff
- Karen Graham
- Roger Howe
- Ken Levasseur
- Steve Maurer
- Cleve Moler
- Stephanie Ragucci
- Glenn Stevens

Advisors

- Ansuman Bagchi
- Nancy Baxter
- David Bressoud
- Gail Burrill
- Guershon Harel
- Dan Teague
- Alan Tucker
- Jim Ward
- Max Warshauer

Field test schools

- Lawrence High School, Lawrence MA
- Mercersburg Academy, PA
- Strath Haven High School, PA
- Andover High School, Andover MA
- Cleveland School of Science and Medicine, OH
- Haverhill High School, Haverhill MA
- San Marcos High School, TX
- Chelsea High School, Chelsea MA
- Albert Einstein High School, MD
- Castilleja School, Palo Alto, CA

Example 2:

A Technology-rich Senior Topics Course

Monthly Payments

Suppose you want to buy a car that costs $10,000. You don't have much money, but you can put $1000 down and pay $350 per month. The interest rate is 5%, and the dealer wants the loan paid off in three years. What kind of car can you buy?

This leads to the question

"How does a bank figure out
the monthly payment on a loan?"
or
"How does a bank figure out
the balance you owe at the end of the month?"

Take 1 (no interest)

What you owe at the end of the month is what you owed at the start of the month minus your monthly payment.

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Take 2

What you owe at the end of the month is what you owed at the start of the month, *plus* $\frac{1}{12}$ *of the yearly interest on that amount*, minus your monthly payment.

$$b(n,m) = \begin{cases} 9000 & \text{if } n = 0 \\ \left(1 + \frac{.05}{12}\right) b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

Students can then use successive approximation to make

$$b(36, ???) = 0$$

Let's try it:

What's the monthly payment

on a loan of \$9000 for 36 months?

Up a Notch

Pick an interest rate and keep it constant. Suppose you want to pay off a car in 24 months. Investigate how the monthly payment changes with the cost of the car:

1. Make a table like this:

Cost of car (in K of dollars)	Monthly payment
10	
11	
12	
13	
14	
15	
⋮	⋮

2. Describe a pattern in the table. Use this pattern to find either a closed form or a recursive rule that lets you calculate the monthly payment in terms of the cost of the car in thousands of dollars. Model your function with your CAS and use the model to to find the monthly payment on a $26000 car. Check your result with the original "b" model

Michelle Connelly

95

a)

x	y(x)	
0	-.3	> 30
1	29.7	> 30
2	59.7	> 30
3	89.7	> 30
4	119.7	> 30
5	149.7	> 30
6	179.7	> 30
7	209.7	> 30
8	239.7	> 30
9	209.7	> 30
10	299.7	> 30
11	329.7	> 30
12	359.7	> 30
13	389.7	> 30
14	419.7	> 30
15	449.7	> 30
16	479.7	> 30
17	509.7	> 30
18	539.7 ✓	

b) $y(x) = \begin{cases} .3 & \text{if } x = 0 \\ y(x-1)+30 & \text{if } x > 0 \end{cases}$ ✓

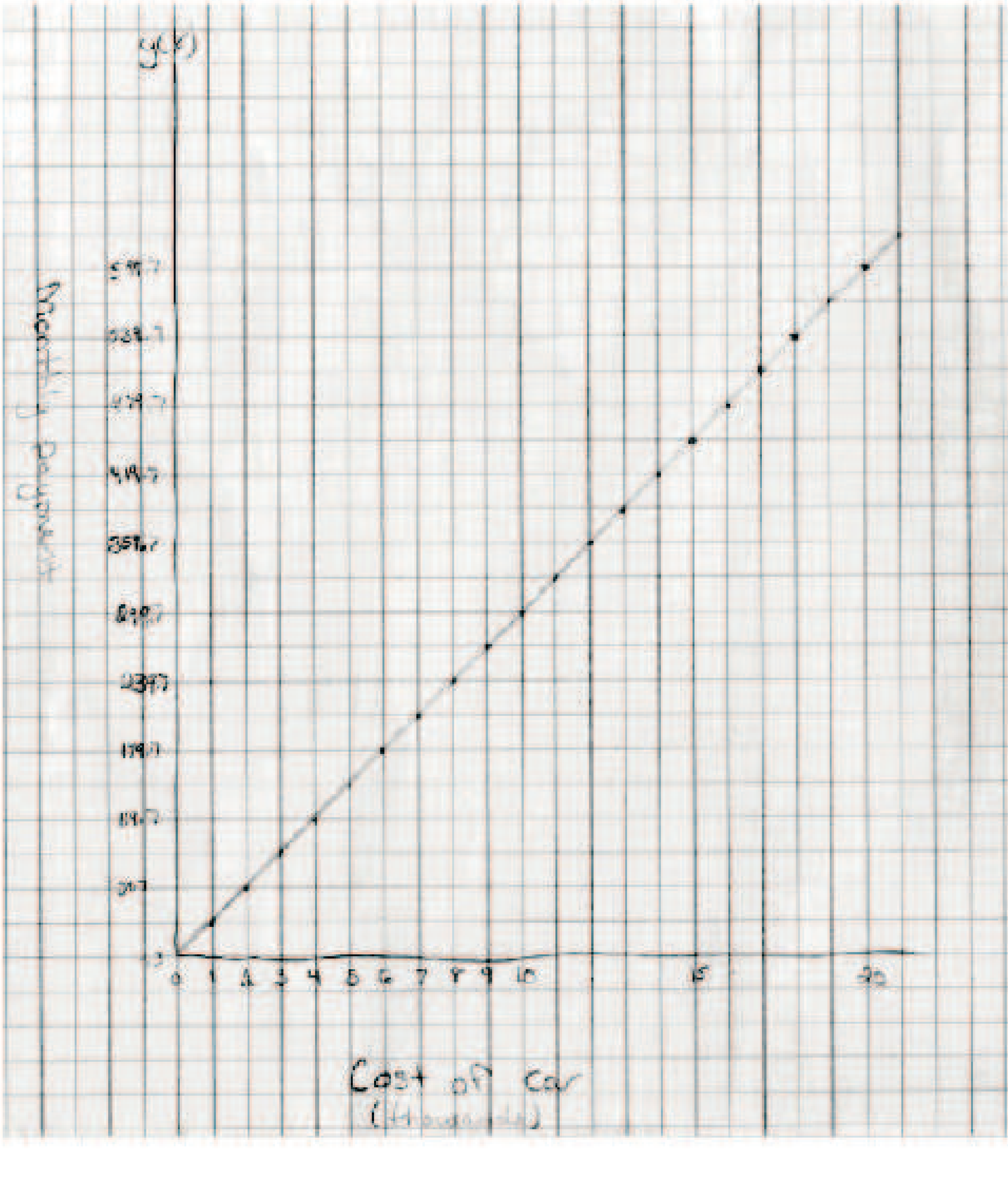
describe pattern ok (on back)

y(x) = ~~[illegible]~~ ?

d) $86000 car $279.7 monthly payment ✓

c) y(x)
Func
if x=0 Then
Return .3
Else
Return y(x-1)+30 ✓

- I changed the amount of the cost of the car then I changed the monthly payment until I found the right monthly payment.
- I found that each time the cost of the car went up $1000 the monthly payment went up $30.



y(x)
Monthly Payments
179.7
0 1 2 3 4 5 6 7 8 9 10
15
20
Cost of car

The balance at the end of 36 months with a monthly payment of $250 can be gotten by entering `b(36,250)` in the calculator:

```
:            b(36,250)
:           764.92
```

⋮

Because we are using a CAS, we can do it *generically*: The balance at the end of 36 months with a monthly payment of m can be gotten by entering `b(36,m)` in the calculator:

```
:               b(36,m)

:              10453.3- 38.7533*m
```

So,

```
:            solve(10453.3- 38.7533*m=0,m)
:            m = 269.738
```

A monthly payment of $269.74 will do the trick.

Suppose you borrow \$12000 at 5% interest. Then you are experimenting with this function:

$$b(n,m) = \begin{cases} 12000 & \text{if } n = 0 \\ (1 + \frac{.05}{12}) \cdot b(n-1,m) - m & n > 0 \end{cases}$$

Notice that

$$1 + \frac{.05}{12} = \frac{12.05}{12}$$

Call this number q. So, the function now looks like:

$$b(n,m) = \begin{cases} 12000 & \text{if } n = 0 \\ q \cdot b(n-1,m) - m & \text{if } n > 0 \end{cases}$$

where q is a constant.

Then at the end of n months, you could unstack the calculation as follows:

$$
\begin{aligned}
b(n,m) &= q \cdot b(n-1,m) - m \\
&= q\,(q \cdot b(n-2,m) - m) - m \\
&\qquad = q^2 \cdot b(n-2,m) - qm - m \\
&= q^2\,(q \cdot b(n-3,m) - m) - qm - m \\
&\qquad = q^3 \cdot b(n-3,m) - q^2 m - qm - m \\
&\qquad \vdots \\
&= q^n \cdot b(0,m) - q^{n-1}m - \ldots - q^2 m - qm - m \\
&= 12000 \cdot q^n - m(q^{n-1} + \ldots + q^2 + q + 1)
\end{aligned}
$$

The last series is geometric; summing it, we get

$$b(n, m) = 12000\, q^n - m\frac{q^n - 1}{q - 1}$$

Setting $b(n, m)$ equal to 0 gives an explicit relationship between m and the cost of the car:

$$m = 12000\, \frac{(q-1)q^n}{q^n - 1}$$

or, in general,

$$\text{monthly payment} = \text{cost of car} \times \frac{(q-1)q^n}{q^n - 1}$$

where n is the term of the loan and

$$q = 1 + \frac{\text{interest rate}}{12}$$