Starting Off Right in Arithmetic

by Roger Howe

"The greatest calamity in the history of science
was the failure of Archimedes
to invent positional notation."

- C. F. Gauss

The Singapore elementary mathematics curriculum has a carefully structured approach to learning the beginnings of whole number base 10 arithmetic. (In fact, this approach is not unique to Singapore; it is a feature of many Asian curricula.) We include in this topic learning the digits, becoming familiar with two-digit numbers, and adding, subtracting and comparing these numbers (including learning the "addition and subtraction facts", that is, knowing the sums of single digit numbers and the corresponding subtractions).

The basic learning sequence followed is as follows:

1) Learning the numbers to 10.

2) Learning the addition and subtraction facts to 10.
This is approached by means of "number bonds", which promotes the idea of thinking of larger numbers as being combinations of smaller numbers, in a triangular relationship. For example, the addition fact $3 + 4 = 7$ would be thought of as saying that 3 and 4 combine to make 7, but also, that if you have 3 and want 7, then you need 4 more, or that 7 is 4 more than 3, or that if you have 7 and remove 4 then you are left with three. It also entails the commutative rule in these cases, so that 7 being composed of 3 and 4 could be written either as $3 + 4 = 7$ or $4 + 3 = 7$. These relationships are mastered thoroughly.

3) Learning the numbers to 20
This is done by describing the teen numbers (including 11 and 12) as 10 plus some 1s. Thus, it includes a first introduction to the concept of place value.

4) Learning the addition and subtraction facts (the sum of any two digits, and the corresponding subtraction equations comprehended by the "number bond" idea).
Here the emphasis is not on memorization, but on understanding how place value, in the sense of topic 3), interacts with addition and subtraction. In particular, addition and subtraction of single digit numbers proceeds via "making (or unmaking) a ten". Thus, to add $6 + 7$, one recognizes that to make 10 from 6, one needs 4 more, so one takes it from the 7, leaving 3, so $6 + 7 = 6 + (4+3) = (6+4) + 3 = 10 + 3 = 13$. (Equally well, one might write (or think) $6 + 7 = (3 + 3) + 7 = 3 + (3 + 7) = 3 + 10 = 13$.) Earlier work on item 2) makes each of these steps very rapid, so that the whole process can be completed fairly quickly, Practice will result in memorization and automatic recall of many of these results, but the basis for reproducing them quickly is already provided by items 2) and 3).
Subtraction is done similarly. To find 13 - 7, one recognizes that one cannot get 7 from 3, so one breaks the 10 into $7 + 3$, takes the 7, leaving 3, which is added to the 3 in the ones place: $13 - 7 = (10 + 3) - 7 = (7 + 3) + 3 - 7 = 3 + 3 = 6$. Alternatively, one can begin by subtracting 3, eliminating the ones place, then subtract the extra 4 from the 10: $13 - 7 = (10 + 3) - 7 = (10 + 3) - (3 + 4) = 10 - 4 = 6$. These processes get students used to "making and unmaking a 10", thus preparing them for two-digit addition and subtraction with renaming.

5) Learning the numbers to 40, with addition and subtraction.
Here the main point is to understand a two-digit number as some 10s and some 1s. Then work with sums shows that, to add, one adds the 10s and the 1s separately. After this principle is established, the case when there are more than ten 1s is considered, and one learns to combine the 10 in the teen number with the other 10s, that is,, to rename or carry. The groundwork for this has already been laid in item 3).

Subtraction is dealt with similarly. When no renaming is needed, one operates on the 10s and on the 1s separately. When this is understood, one then deals with the situation when a 10 must be decomposed in order to compute the result. Again, the basis for this has been laid in item 3).

6) Numbers to 100, with addition and subtraction.

This is a continuation of item 5) . The discussion of two-digit numbers is divided into two stages mainly in order to allow students to adjust gradually to larger numbers. Also, when adding numbers less than 40, the possibility of going over 100 never arises.

7) Comparison (ordering).

Comparison has in fact been going on throughout the items 1) through 6). Numbers up to 10 are compared when they are introduced. Likewise, the teen numbers are compared when they are introduced. The main idea when comparing two general two-digit numbers is that the number of 10s is decisive, exccept when both numbers have the same 10s digit, in which case one looks at the 1s digit.

Supplementary remarks:

i) Pedagogy: The above outline is predominantly about mathematical issues. How the topics are presented is also important. The Singapore texts follow a consistent progression of "concrete - pictorial - abstract". This entails first presenting mathematical ideas in richly illustrated settings, then introducing a schematic framework for thinking about them, and finally moving to a fully symbolic treatment.

ii) Understanding the operations; word problems: The discussion above covers only the issues connected with base 10 notation. For a robust mastery of arithmetic, students must also have well-developed conceptions of what addition and subtraction mean. Mathematics educators have developed a taxonomy of addition and (mainly) subtraction problems comprising ten categories of problems ([Cetal]). Students should see word problems from most of these categories in the early stages learning addition and subtraction. Each type of problem can be posed with larger numbers as students become more adept in base ten arithmetic. Understanding of the operations can also be developed and assessed by assigning "inverse word problems" – giving an equation, perhaps with a blank to fill in, and asking students to create a word problem for which that equation is the numerical formulation.

iii) Coordination with measurement: It is desirable that students begin learning early the connections between arithmetic and measurement, especially linear measurement. This involves associating numbers with bars of corresponding lengths, and seeing that addition corresponds to laying bars end to end, and subtracting corresponds to finding a bar whose length is the difference between bars. Work with bars can be used to reinforce place value ideas by using bars of length 10 along with unit cubes to represent two digit numbers. The idea of "adding the tens and adding the ones: can be represented nicely with such manipulatives.

Later on, these techniques can be coordinated with the number line. The number line is particularly useful for visualizing comparisons, and also for giving a geometric representation of the rearrangements implicit in the "adding the tens and adding the ones" strategy.

Although it is common in extending such work to larger numbers to represent 100 as a "flat" - a square consisting of 100 unit cubes, and 1000 as a cube, I feel that it is important for students to see, and be reminded periodically, what a length 100 units long and even 1000 units long is like. This helps to underscore how rapidly the values associated to successive decimal places increase, and also to keep students mindful that larger numbers still live on the (number) line. (To actually produce a bar representing 10,000 is probably impractical (and in any case, 10,000 is not usually encountered in first grade), but when 10,000 and larger base ten units are addressed, a class discussion of how long such a bar would be could be valuable.)

[Cetal] Children's Mathematics, by T. Carpenter, E. Fennema, M. Franke and L. Levi; Heinemann Press, 1999.