The Power of Modular Arithmetic
Exploring the Structure of $\mathbb{Z}/7$

1. Perform the following operations modulo 7:

   \[
   \begin{align*}
   3 + 2 & \quad 5 + 4 & \quad 5 - 1 & \quad 1 - 5 & \quad 3 \cdot 2 & \quad 4 \cdot 6
   \end{align*}
   \]

2. Answer the following sets of questions.

   (a) What is the remainder when 22 is divided by 7? What is the remainder when 8 is divided
       by 7? What is the remainder when $22 \cdot 8$ is divided by 7?

   (b) What is the remainder when 10 is divided by 7? What is the remainder when 16 is divided
       by 7? What is the remainder when $10 \cdot 16$ is divided by 7?

   (c) What is the remainder when 102 is divided by 7? What is the remainder when 104 is divided
       by 7? What is the remainder when $102 \cdot 104$ is divided by 7?

3. Make a conjecture based on parts (a) through (c) of the previous problem.

4. Complete the following multiplication table for the number system $\mathbb{Z}/7$. (The table here is small;
   you will probably want to make a table on your own paper.) Try to find as many patterns in
   the table as you can; these will help you complete the table more quickly.

   \[
   \begin{array}{c|ccccccc}
   \times & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
   \hline
   0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
   1 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
   2 & 2 & 4 & 6 & 1 & 3 & 5 & 0 \\
   3 & 3 & 6 & 2 & 5 & 1 & 4 & 0 \\
   4 & 4 & 1 & 5 & 2 & 6 & 3 & 0 \\
   5 & 5 & 3 & 1 & 6 & 4 & 2 & 0 \\
   6 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
   \end{array}
   \]

5. Suppose we wanted to talk about the number $\frac{1}{3}$ in the number system $\mathbb{Z}/7$. Which number in
   $\mathbb{Z}/7$ is this? What about the number $\frac{1}{3}$? Is the sum $\frac{1}{2} + \frac{1}{3}$ in $\mathbb{Z}/7$ equal to $\frac{5}{6}$?

6. Which of the following can be evaluated in $\mathbb{Z}/7$?

   \[
   \sqrt{4} \quad \sqrt{3} \quad \sqrt{2}
   \]

7. Evaluate each of the following in $\mathbb{Z}/7$: $0^6$, $1^6$, $2^6$, $3^6$, $4^6$, $5^6$, $6^6$. What do you notice?

8. Evaluate $5^{2471}$ modulo 7.