Section 7.4: More Factoring and Graphing

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Warm-Up

Please work on the following problems as a warm-up:

1. Multiply:
   \[(x^2 - 2x + 5)(x - 3)\]

2. Factor completely:
   \[6x^2 + 19x - 20\]

3. Factor completely using the difference of two squares:
   \[k^7 - k^3\]

4. Multiply:
   \[(x + y)(x^2 - xy + y^2)\]
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Please work on the following problems as a warm-up:

1. **Multiply:**
   \[(x^2 - 2x + 5)(x - 3) = x^3 - 5x^2 + 11x - 15\]

2. **Factor completely:**
   \[6x^2 + 19x - 20\]

3. **Factor completely using the difference of two squares:**
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4. **Multiply:**
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   \[k^7 - k^3 = k^3(k^2 + 1)(k - 1)(k + 1)\]

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   \[(x + y)(x^2 - xy + y^2) = x^3 + y^3\]
Factoring the Sum of Cubes

In the warm-up problems, we showed that

\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2). \]

We can think of this as a rule for factoring, and use it when we need to factor the sum of two cubes. For example, factor

\[ x^3 + 8 \]
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We can think of this as a rule for factoring, and use it when we need to factor the sum of two cubes. For example, factor

\[ x^3 + 8 = (x + 2)(x^2 - 2x + 4) \]
We now know that
\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2). \]

Can you find an equivalent expression for \( x^3 - y^3 \)?

Hint:
\[ x^3 - y^3 = x^3 + (-y)^3 =? \]
Factoring a Difference of Two Cubes

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Hint:

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\begin{align*}
    x^3 - y^3 &= x^3 + (-y)^3 \\
    &= (x + (-y))(x^2 - x(-y) + (-y)^2)
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\begin{align*}
    x^3 - y^3 &= x^3 + (-y)^3 \\
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             &= (x - y)(x^2 + xy + y^2)
\end{align*}
\]
Factoring a Sum or Difference of Two Cubes

Write these equations in your notes: you’ll need them!

\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2) \]
\[ x^3 - y^3 = (x - y)(x^2 + xy + y^2) \]
Factoring a Sum or Difference of Two Cubes: Examples

\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2) \]
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Factor the polynomial completely:

1. \[ c^3 - 729 \]
2. \[ p^3 + w^{12} \]
3. \[ 3c^4 - 81c \]
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Factor the polynomial completely:

1. \[ c^3 - 729 = (c - 9)(c^2 + 9c + 81) \]
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\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2) \]
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Factor the polynomial completely:

1. \[ c^3 - 729 = (c - 9)(c^2 + 9c + 81) \]
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Factor the polynomial completely:

1. \[ c^3 - 729 = (c - 9)(c^2 + 9c + 81) \]
2. \[ p^3 + w^{12} = (w^4 + p)(w^8 - pw^4 + p^2) \]
3. \[ 3c^4 - 81c = 3c(c - 3)(c^2 + 3c + 9) \]
Roots and Factors of Quadratics

Determine the roots of the quadratic functions below, then factor them.

Remember, \( a \) is a root of the function \( f(x) \) if and only if \( f(a) = 0 \).

1. \( p(x) = x^2 - 7x + 12 \)
   - Factors: \((x - 3)(x - 4)\)
   - Roots: 3 and 4

2. \( g(x) = x^2 + 3x - 10 \)
   - Factors: \((x + 5)(x - 2)\)
   - Roots: -5 and 2

3. \( m(x) = 6x^2 + 19x - 20 \)
   - Factors: \((6x - 5)(x + 4)\)
   - Roots: \(\frac{5}{6}\) and -4

Can you state a relationship between the factors of a quadratic and the roots?
Roots and Factors of Quadratics

Determine the roots of the quadratic functions below, then factor them.

Remember, a is a root of the function $f(x)$ if and only if $f(a) = 0$.

1. $p(x) = x^2 - 7x + 12 = (x - 3)(x - 4)$. $p(x)$ has roots 3 and 4.

2. $g(x) = x^2 + 3x - 10$

3. $m(x) = 6x^2 + 19x - 20$
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3. \( m(x) = 6x^2 + 19x - 20 = (6x - 5)(x + 4) \). \( m(x) \) has roots \( \frac{5}{6} \) and -4.

Can you state a relationship between the factors of a quadratic and the roots?
If $a$ is a root of the polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$.

Example: Factor the polynomial $f(x) = x^2 - 21x + 68$.

First, find the roots of $f(x)$:
If $a$ is a root of the polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$.

Example: Factor the polynomial $f(x) = x^2 - 21x + 68$.

First, find the roots of $f(x)$: 17 and 4.

Thus, $f(x)$ must have two factors:
If \( a \) is a root of the polynomial \( p(x) \), then \( (x - a) \) is a factor of \( p(x) \).

Example: Factor the polynomial \( f(x) = x^2 - 21x + 68 \).

First, find the roots of \( f(x) \): 17 and 4.

Thus, \( f(x) \) must have two factors: \((x - 17)\) and \((x - 4)\).

Let’s guess that \( f(x) = (x - 17)(x - 4) \). Confirm this factoring is correct by expanding the terms.
If \( a \) is a root of the polynomial \( p(x) \), then \((x - a)\) is a factor of \( p(x) \).

Example: Factor the polynomial \( f(x) = 3x^2 - 16x + 13 \).

First, find the roots of \( f(x) \):
If \( a \) is a root of the polynomial \( p(x) \), then \( (x - a) \) is a factor of \( p(x) \).

Example: Factor the polynomial \( f(x) = 3x^2 - 16x + 13 \).

First, find the roots of \( f(x) \): 1 and \( \frac{13}{3} \).

Thus, \( f(x) \) must have two factors:
If a is a root of the polynomial \( p(x) \), then \( (x - a) \) is a factor of \( p(x) \).

Example: Factor the polynomial \( f(x) = 3x^2 - 16x + 13 \).

First, find the roots of \( f(x) \): 1 and \( \frac{13}{3} \).

Thus, \( f(x) \) must have two factors: \( (x - 1) \) and \( (x - \frac{13}{3}) \).

Let’s guess that \( f(x) = (x - 1)(x - \frac{13}{3}) \). Show that this factoring is not correct! What can we do to fix it?
If $a$ is a root of the polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$.

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$$(x - 1)(x - \frac{13}{3}) = x^2 - \frac{16}{3}x + \frac{13}{3} \neq 3x^2 - 16x + 13.$$
If \( a \) is a root of the polynomial \( p(x) \), then \((x - a)\) is a factor of \( p(x) \).

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\[(x - 1)(x - \frac{13}{3}) = x^2 - \frac{16}{3}x + \frac{13}{3} \neq 3x^2 - 16x + 13.\] If we multiply by 3, the coefficient of the leading term, everything does work:

\[f(x) = 3(x - 1) \left( x - \frac{13}{3} \right) = (x - 1)(3x - 13).\]
To factor a polynomial $p(x)$ with leading coefficient $c$ you can:

1. Find the roots of $p(x)$. Let’s call them $a_0, a_1, \ldots, a_n$.
2. Then $p(x) = c(x - a_0)(x - a_1)\ldots(x - a_n)$.

Examples: Factor completely:

1. $f(x) = 20x^2 + 89x + 18$
2. $f(x) = x^2 + 1$
3. $f(x) = x^2 - 2$