Card Shuffling and Repeated Decimals
A Math Circle activity inspired by Steve Pelikan and Steve Phelps

A “perfect shuffle” means “divide the deck of cards into two equal piles of cards, and exactly interleave the two decks keeping the top card on top.” For an ordered deck of 52 cards can we restore the order by performing enough “perfect shuffles”?

1 Warm up

1. Start with an 8-card deck arranged in order 1-8. Perform a “perfect shuffle” on this 8-card deck. How does the order of the cards change? How many “perfect shuffles” need to be performed to restore the order of the deck of 8 cards?

2. Arithmetic in mod 10 is fun and easy because you only use the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. After any calculation you only report the remainder after dividing by 10. For example, in mod 10, 6 + 5 = 1 because 1 is the remainder when 6 + 5 is divided by 10.

Answer the these questions in mod 10.

(a) $2 + 2 = \underline{0}$
(b) $3 \cdot 4 = \underline{2}$
(c) $\underline{2} + 5 = 2$
(d) $4 \cdot \underline{2} = 2$
(e) $7 \cdot \underline{5} = 4$
(f) $\underline{3}^3 = 6$

3. Now answer the same questions in mod 7.

(a) $2 + 2 = \underline{0}$
(b) $3 \cdot 4 = \underline{2}$
(c) $\underline{2} + 5 = 2$
(d) $4 \cdot \underline{2} = 2$
(e) $7 \cdot \underline{5} = 4$
(f) $\underline{3}^3 = 6$

3. (a) Find the repeating decimal for $\frac{1}{11}$. List all the remainders you encountered during the long division, starting with 1 and 10.

(b) Find the repeating decimal for $\frac{18}{37}$. List all the remainders you encountered, including 1 and 10.

(c) Find the repeating decimal for $\frac{1}{37}$. List those remainders!
4. What do these equations have to do with \( \frac{1}{41} \)? Fill in any blanks with the correct information.

\[
\begin{align*}
1 &= 0 \cdot 41 + 1 \\
10 &= 0 \cdot 41 + 10 \\
100 &= 2 \cdot 41 + 18 \\
180 &= 4 \cdot 41 + 16 \\
160 &= 3 \cdot 41 + ____ \\
____ &= 9 \cdot 41 + 1
\end{align*}
\]

5. Complete this table. Splitting up the work is a great idea. Think carefully about what mathematical shortcuts you could use to work less.

<table>
<thead>
<tr>
<th>n</th>
<th>Powers of 10 in ( mod \ n )</th>
<th>Cycle length</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1, 10, 9, 12, 3, 4, 1, ...</td>
<td>6</td>
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<tr>
<td>3</td>
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<td>41</td>
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<tr>
<td>51</td>
<td>1, 10, 49, 31, 4, 40, 43, 22, 16, 7, 19, 37, , , , , , ...</td>
<td>16</td>
</tr>
</tbody>
</table>

6. What do the powers of 10 in \( mod \ n \) have to do with the decimal expansion of \( \frac{1}{n} \)?

7. What do these equations have to do with the base 2 decimal expansion for \( \frac{1}{21} \)? Fill in the blanks.

\[
\begin{align*}
1 &= 0 \cdot 21 + 1 \\
2 &= 0 \cdot 21 + 2 \\
4 &= 0 \cdot 21 + ____ \\
8 &= ____ \cdot 21 + 8 \\
16 &= 0 \cdot 21 + ____ \\
32 &= 1 \cdot 21 + \\
22 &= 1 \cdot 21 + ____ \\
\end{align*}
\]
8. Complete this table. Again, splitting up the work is a great idea. Think carefully about what mathematical shortcuts you could use to work less.

<table>
<thead>
<tr>
<th>n</th>
<th>Powers of 2 in mod n</th>
<th>Cycle length</th>
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</thead>
<tbody>
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<tr>
<td>19</td>
<td>1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, ...</td>
<td>18</td>
</tr>
</tbody>
</table>

9. Take a deck of 10 cards. Put them in order 1-10. Write out the order of the cards after each “perfect shuffles”. Repeat this until the cards are back in order. Do you see any relations between the “perfect shuffles” of your deck of 10-cards and the table of information above?

10. How can we determine how many “perfect shuffles” we need to do to restore order to a deck of 52 cards? Explain why this will work. Test your number on a deck of 52 cards with a partner.