The Seven Bridges of Königsberg

The city of Kaliningrad, formally called Königsberg, is located in Russia. The river Pregel divides the city in two and contains two islands. Back in the 1700’s there were seven bridges connecting the islands to the rest of the city and to each other, as shown in the picture.

The citizens of the city posed a question: *Is it possible to find a walk through the city that crosses each bridge exactly once?*

Notice that the paths taken on land are not important. Because of this we can represent the city as a graph, where the islands are *vertices* and the bridges are *edges*. The *degree* of a vertex is the number of edges it has. In graph theory, the desired walk is called an *Euler walk*. 
Let’s consider some simpler examples. Answer the following questions:

1. What are the possible walks for the following graphs?
2. What do you notice about the your starting and ending vertices?
3. If you start at certain vertices, it is not possible to perform a walk. What do you notice about these vertices?

Fill in the missing numbers: Other than _____ vertices, every time you “visit” a vertex this corresponds to _____ edges.
Conjecture: What conditions does a graph need in order to have an Euler walk?

Suppose an extra bridge is added. Is an Euler walk then possible? Does it matter where the bridge is added?

An Euler walk that ends at the same vertex it begins is called an Eulerian circuit. What conditions are necessary for a graph to have an Eulerian circuit?

A “different” problem

Is it possible to draw a continuous curve that passes through each of the ten edges of the following figure exactly once? (A curve that passes through a vertex is not allowed.)
A variation with identified nodes

Three friends Peter, Ivan, and Daniel all work in the evenings as delivery boys at the local pizza, ice cream, and donut shop, respectively. Each day they must deliver their products to each of the different pieces of land and then end at the bank to deposit the money they’ve made. All three of them claim to have been able to walk all the bridges without repeating. But it’s dark when they are on their delivery routes and none of them have been able to repeat the feat during the day.

1. One day at the bank Daniel makes fun of Peter because he made more tips than Peter did. Peter gets upset and wants to get even. Peter, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin at the pizza shop, walk the bridges, and end at the bank. But, he doesn’t want Daniel to duplicate the feat from the donut shop. Where does Peter build the eighth bridge?

2. Daniel, jealous of Peter’s accomplishment, wants to build a ninth bridge, enabling him to begin at the donut shop, walk the bridges, and end at the bank. In addition he wants to the build the bridge so that Peter is no longer able to walk the bridges starting at the pizza shop as before. Where does Daniel build the ninth bridge?

3. Ivan, worried that this competition is ruining the three’s friendship wants to build a tenth bridge that allows all three to walk the bridges. Instead of ending at the bank, he wants everyone to be able to walk the bridges so that they end at their respective shops, thus decreasing bragging time. Where does Ivan build the tenth bridge?