SNOWFLAKES AND THE GEOMETRIC SERIES

1. The Geometric series

The finite geometric series formula says that if \( a \) is any integer different than 1 or 0 then we have

\[
1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}.
\]

The infinite geometric series formula says that if \( a \) is a number with \( |a| < 1 \) then

\[
1 + a + a^2 + \cdots = \frac{1}{1 - a}.
\]

**Problem 0** Show that the finite geometric series formula is correct. Use the finite geometric series formula to convince yourself that the infinite geometric series formula is correct.

The two formulas above have lots of uses and are very important. We hope that the problems below will convince you of this.

2. The Area of a Snowflake

We will construct two complicated geometric sets and try to find their areas and perimeters.

We start with the inner snowflake. To construct the inner snowflake, we first construct a family of polygons, \( I_1, I_2, I_3, \ldots \) as follows:

- \( I_1 \) is an equilateral triangle.
- \( I_2 \) is obtained from \( I_1 \) by adding an equilateral triangle to the middle third of each side of \( I_1 \).
- \( I_3 \) is obtained from \( I_2 \) by adding an equilateral triangle to the middle third of each side of \( I_2 \).
  
  In general, \( I_{n+1} \) is obtained from \( I_n \) by adding an equilateral triangle to the middle third of each side of \( I_n \).

The inner snowflake \( S_I \) consists of all points that belong to some \( I_n \).

The second set that we construct is the outer snowflake. To construct it, we start by making a family of polygons, \( O_1, O_2, O_3, \ldots \) as follows:

- \( O_1 \) is a regular hexagon.
- \( O_2 \) is obtained from \( O_1 \) by removing an equilateral triangle from the middle third of each side of \( O_1 \).
- \( O_3 \) is obtained from \( O_2 \) by removing an equilateral triangle from the middle third of each side of \( O_2 \).
  
  In general, \( O_{n+1} \) is obtained from \( O_n \) by removing an equilateral triangle from the middle third of each side of \( O_n \).

The outer snowflake \( S_O \) consists of all points that belong to all \( O_n \).

Before we continue we will need some results from geometry.
**Problem 1** Let $\triangle BAC$ be an isosceles triangle with $\angle BAC = 120^\circ$. Let $E, F$ be the points that trisect $BC$. Then $\triangle AEF$ is an equilateral triangle, and the two triangles $\triangle AEB$ and $\triangle AFC$ are congruent isosceles triangles with $120^\circ$ angles.

**Problem 2** Show the following two things:

- If $T$ is an equilateral triangle with side of length $a$, then the altitude of $T$ has length $\frac{a\sqrt{3}}{2}$, and the area of $T$ is $\frac{\sqrt{3}}{4}a^2$.
- If $R$ is an isosceles triangle with $120^\circ$ angle with two sides of length $a$ then the third side of $R$ has length $a\sqrt{3}$.

We now construct $I_1, I_2, I_3, \ldots$ and $O_1, O_2, O_3, \ldots$ such that

$I_1 \subset I_2 \subset I_3 \subset \cdots \subset O_3 \subset O_2 \subset O_1$.

Let $O_1$ be a regular hexagon with side 1 and let $I_1$ be an equilateral triangle inscribed in $O_1$. Then $O_1 \setminus I_1$ consists of three isosceles $120^\circ$ triangles with short side 1. Then by **Problem 1** the sides of $I_1$ have length $\sqrt{3}$.

Our general procedure for constructing polygons was already described above. For each $n$, $O_n \setminus I_n$ will consists of a family of congruent isosceles $120^\circ$ triangles and $O_{n+1} \setminus I_{n+1}$ is obtained from $O_n \setminus I_n$ by removing an equilateral triangle from the middle third of each side of each isosceles $120^\circ$ triangles. Notice that **Problem 2** guarantees that thus process obtains from a set of isosceles $120^\circ$ triangles to a new set of $120^\circ$ triangles. See the next page for a picture.

Let

$s_n = \text{length of a side of } I_n$,
$t_n = \text{area of equilateral triangle with side } s_n$,
$m_n = \text{number of sides of } I_n$,
$a_n = \text{area of } I_n$,
$S_n = \text{length of a side of } O_n$,
$T_n = \text{area of equilateral triangle with side } S_n$,
$M_n = \text{number of sides of } O_n$,
$A_n = \text{area of } O_n$.

Notice that we have the following formulas:

\[
s_{n+1} = \frac{1}{3}s_n,
\]
\[
m_{n+1} = 4m_n,
\]
\[
a_{n+1} = a_n + m_n t_{n+1},
\]
\[
S_{n+1} = \frac{1}{3}S_n,
\]
\[
M_{n+1} = 4M_n,
\]
\[
A_{n+1} = A_n - M_n T_{n+1}.
\]

Since an equilateral triangle of side $s$ can be decomposed into nine equilateral triangles of side $\frac{s}{3}$ we have

\[
t_{n+1} = \frac{t_n}{9}, \quad \text{and} \quad T_{n+1} = \frac{T_n}{9}.
\]

Also

\[
a_1 = \text{area}(I_1) = t_1.
\]
Problem 3 Why do we have $A_1 = 6T_1$?

Problem 4 Show that area($I_1$) = $\frac{1}{2}$area($O_1$).

Problem 5 Make a table of values of $m_n$, $t_n$, $M_n$, $T_n$, $M_n-1T_n$.

Problem 6 Using the table write down formulas for area($I_{n+1}$) and area($O_{n+1}$) and show that both the inner snowflake and the outer snowflake have area $\frac{4}{5}$.

Problem 7 Calculate the length of the perimeter of $I_n$. Calculate the length of the perimeter of $O_n$.

Problem 8 Show that the inner snowflake is not equal to the outer snowflake.

3. Other Problems

Problem 9 Use the geometric series formula to show that $0.99999999\ldots = 1$.

Problem 10 Find the following sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Problem 11 The people in the ancient city of Babylon decided to build a tower by stacking infinitely many cubes on top of each other. The first cube is to have side 1 foot, the second cube is to have side $\frac{1}{\sqrt{2}}$ feet, the third cube $\frac{1}{\sqrt{3}}$ feet and so on. Do you think that they will be able to build this tower?

Problem 12 Let us consider arithmetic modulo 5. That is let us think of the five numbers 0, 1, 2, 3, 4 and let us define addition and multiplication on these numbers by using the usual addition and multiplication and then taking the remainder of dividing by 5. Let us use symbols $\oplus$ and $\otimes$ for these operations. For example,

$$3 + 4 = 7, \text{ but the remainder of 7 when I divide by 5 is 2 therefore } 3 \oplus 4 = 2,$$

$$4 \cdot 4 = 16, \text{ but the remainder of 16 when I divide by 5 is 1 therefore } 4 \otimes 4 = 1.$$

Compute $2 \oplus 2, 4 \oplus 1, 3 \oplus 3, 0 \oplus 4$. In fact, complete the following multiplicative table for system for arithmetic modulo 5:

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<th>0</th>
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We will use power notation as follows $a^{\otimes 0} = 1$, $a^{\otimes 1} = a$, $a^{\otimes 2} = a \otimes a$, $a^{\otimes 3} = a \otimes a \otimes a$ and so on. Show that $1 = 2^{\otimes 4}$, $2 = 2^{\otimes 1}$, $3 = 2^{\otimes 3}$, $4 = 2^{\otimes 2}$. Use this and the finite geometric series formula to show that

$$1 \oplus 2 \oplus 3 \oplus 4 = 0.$$

Similarly, show that

$$1^{\oplus 2} \oplus 2^{\oplus 2} \oplus 3^{\oplus 2} \oplus 4^{\oplus 2} = 0.$$

Do this for exponents $\otimes 3, \otimes 4, \otimes 5, \otimes 6, \otimes 7, \otimes 8$ and guess a general formula.

Problem 13 Use arithmetic module 5 to show that

$$11^{111} + 12^{111} + 13^{111} + 14^{111}$$

is a multiple of 5.