**Pick’s Theorem**

*Pick’s Theorem* provides a method for determining the area of a simple polygon whose vertices lie on lattice points of a square grid. This theorem relates the area of a polygon based on the number of *interior points* \((I)\) and *perimeter points* \((P)\).

A *simple polygon* is a polygon that has no holes and no intersecting edges.

**Example:**

The following polygon has:

\[
\begin{align*}
\text{_____ interior points} \\
\text{& _____ perimeter points.}
\end{align*}
\]

Assume each square has an area of 1 unit\(^2\).
The area of the polygon is________.

**What is Pick’s Theorem:**

1. Draw a few polygons with \(I=0\) and various numbers of perimeter points and find their areas. How does changing the perimeter points change the area?

2. Draw a few polygons with a fixed number of perimeter points and try various numbers of interior points. What are their areas? How does changing the number of perimeter points change the area?
Conjecture:

3. The area of a simple polygon with $P$ perimeter points and $I$ interior points is given by:

Exercises:

4. Is it possible to draw an equilateral triangle on a square grid? Give an example or prove that it is not possible.

*Hint: Consider the equation for the area of a triangle and Pythagorean’s Theorem.*

5. Prove that if you throw a $n \times n$ square on a uniform grid, it cannot cover more than $(n + 1)^2$ grid points.

*Hint: First consider how many grid points you have if the edges of the square fall directly on the grid points.*
The **mediant** of any two fraction \( \frac{m_1}{n_1} \) and \( \frac{m_2}{n_2} \) is defined to by \( \frac{\frac{m_1}{n_1} \oplus \frac{m_2}{n_2}}{\frac{n_1}{n_1} + \frac{n_2}{n_2}} = \frac{m_1 + m_2}{n_1 + n_2} \).

The **Farey Series** \( F_N \) is the set of all fractions in lowest terms between 0 and 1 whose denominators do not exceed N, arranged in order of magnitude.

**Example:**
The Farey Series for N=4 is given by:
0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1.

We can construct a Farey Series using mediants. The middle term between any three consecutive terms is the mediant of the two outer terms.

*There are two easy approaches to constructing the Farey Series:*

I. Constructing a “tree” using
mediants:

II. Using “rays”: Draw “rays” coming from the x-axis into the first quadrant. For each ray, stop at the first grid point you reach. Stop when you reach the point (1,1). The order of the rays is same as the order for the Farey Series.
6. Determine the Farey Series for N=6 using the two methods.
7. The Farey Series has the property that for any two consecutive terms \( \frac{m_1}{n_1} \) and \( \frac{m_2}{n_2} \) the following equality holds: \( m_2n_1 - m_1n_2 = 1 \).

We will use Pick’s Theorem to prove that this is true. The following are some facts that we will use:

1. The area of the triangle formed by the points \((0,0), (n_1, m_1), \) and \((n_2, m_2)\) is given by \( 1/2(m_2n_1 - m_1n_2) \).
2. For \( n \neq 0 \) and \( m \neq 0 \), the line connecting \((0,0)\) to the point \((n,m)\) is does not contain any grid points iff \( \gcd(n,m) = 1 \).

Hence, we want to show that the area of the triangle formed by the points \((0,0), (n_1, m_1), \) and \((n_2, m_2)\) is \( \frac{1}{2} \) using Pick’s Theorem.

_Hint: Determine the number of lattice points in and on a single triangle. Justify your reasoning._