We will go through two examples of solving Sturm-Liouville problems. The first will have exact closed form solutions because the boundary conditions are very simple, the second will not have closed form solutions, and will need to be solved graphically and/or numerically.

The first problem is $-y'' = \lambda y$ with $y'(0) = 0$ and $y(1) = 0$. Since this is a Sturm-Liouville problem, we know that the only possible values of $\lambda$ with non-zero solutions are for real $\lambda$’s. The three cases are:

$\lambda = 0$: $y'' = 0$ implies that $y = Ax + B$.

$\lambda < 0$: In this case we can write $y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$.

$\lambda > 0$: In this case we can write $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$.

Now we need to see how the boundary conditions lead to restrictions on the possible values of $\lambda$, the eigenvalues, and of the functions $y(x)$, the eigenfunctions.

In the case $\lambda = 0$, when we impose the boundary conditions we get

$y'(0) = A = 0$, and $y(1) = A + B = 0$

It is easy to see that this requires $A = B = 0$, and has no non-trivial solution. I.e. $\lambda = 0$ is not an eigenvalue. However, we can also make this into a determinant condition by rewriting the pair of equations in matrix form as

$$
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

This has only the trivial solution because the determinant of the matrix is not zero.

Let’s look at the more complicated case where $\lambda < 0$. For convenience, let $\lambda = -\nu^2$ with $\nu > 0$. That way we have $y = Ae^{\nu x} + Be^{-\nu x}$. When we impose the two boundary conditions we get

$y'(0) = \nu A - \nu B = 0$, and $y(1) = Ae^\nu + Be^{-\nu} = 0$.

In matrix form this is

$$
\begin{pmatrix}
\nu & -\nu \\
\nu e^\nu & -\nu e^{-\nu}
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

In this case the determinant is $\nu(e^\nu + e^{-\nu})$. Since we have assumed that $\nu > 0$ and the exponential of any real argument is positive, this can never be zero. That means the only solutions are trivial. Negative $\lambda$’s cannot be eigenvalues.

For $\lambda > 0$ it is convenient to set $\lambda = k^2$ with $k > 0$ so that $y = A\cos(kx) + B\sin(kx)$. The boundary conditions give us

$y'(0) = kB = 0$, and $y(1) = A\cos(k) + B\sin(k) = 0$. 

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In matrix form this becomes
\[
\begin{pmatrix}
0 & k \\
\cos(k) & \sin(k)
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
To have a nonzero solution, we need the determinant to vanish
\[
k \cos(k) = 0.
\]
Since we have assumed \( k > 0 \), that means \( \cos(k) = 0 \) and so \( k \) is an odd multiple of \( \pi/2 \).

To find the eigenfunctions we go back to the matrix. When we row reduce, the second row will be eliminated (since the rank is less than 2), and the solutions will be gotten from just the first row. \( k_n B = 0 \), so the eigenfunctions will be \( y_n = \cos(k_n x) \) with \( k_n = \pi i/2 + n\pi \), \( n = 0, 1, 2, \ldots \), and the eigenvalues are \( \lambda = k_n^2 = ((2n + 1)\pi/2)^2 \).

We really did not need the power of the matrix formulation to work with these, however, in the next example you will see how it helps. I have deliberately used very similar language and structure to emphasize the way these problems can always be solved.

Consider the Sturm-Liouville problem \(-y'' = \lambda y\) where \( y = y(x) \) with boundary conditions \( y(0) + y'(0) = 0 \), and \( y(1) - y'(1) = 0 \). This is a more complicated set of boundary conditions than our earlier examples, but we can approach it using the same ideas.

Since this is a Sturm-Liouville problem, we know that the only possible values of \( \lambda \) with non-zero solutions are for real \( \lambda \)'s. The three cases are:

- \( \lambda = 0 \): \( y'' = 0 \) implies that \( y = Ax + B \).
- \( \lambda < 0 \): In this case we can write \( y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \).
- \( \lambda > 0 \): In this case we can write \( y = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x) \).

Now we need to see how the boundary conditions lead to restrictions on the possible values of \( \lambda \), the eigenvalues, and of the functions \( y(x) \), the eigenfunctions.

In the case \( \lambda = 0 \), when we impose the two boundary conditions we get
\[
y(0) + y'(0) = A + B = 0, \quad \text{and} \quad y(1) - y'(1) = (A + B) - A = B = 0.
\]
It is easy to see that the only solution is \( A = B = 0 \), but, we could also have formulated this in matrix form as
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
Since the matrix in the equation has a nonzero determinant, the only solution is \( A = B = 0 \), the trivial one.

Let’s look at the more complicated case where \( \lambda < 0 \). For convenience, let \( \lambda = -\nu^2 \) with \( \nu > 0 \). That way we have \( y = Ae^{\nu x} + Be^{-\nu x} \). When we impose the two boundary conditions we get
\[
y(0) + y'(0) = (A + B) + \nu(A - B) = (1 + \nu)A + (1 - \nu)B = 0, \quad \text{and} \quad y(1) - y'(1) = Ae^{\nu} + Be^{-\nu} - (\nu Ae^{\nu} - \nu Be^{-\nu}) = (1 - \nu)e^{\nu}A + (1 + \nu)e^{-\nu}B = 0.
\]
In matrix form this is
\[
\begin{pmatrix}
1 + \nu & 1 - \nu \\
(1 - \nu)e^\nu & (1 + \nu)e^{-\nu}
\end{pmatrix}
\begin{pmatrix} A \\ B \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
It will have only the trivial zero solution unless
\[
\det\begin{pmatrix}
1 + \nu & 1 - \nu \\
(1 - \nu)e^\nu & (1 + \nu)e^{-\nu}
\end{pmatrix}
= (1 + \nu)^2 e^{-\nu} - (1 - \nu)^2 e^\nu = 0,
\]
or equivalently
\[
e^\nu = \left| \frac{1 + \nu}{1 - \nu} \right|.
\]

The plot above shows $e^\nu$ in blue, and $\frac{1 + \nu}{1 - \nu}$ in red. There is an intersection at $\nu = 0$, but we are restricted to the case $\nu > 0$ by our assumptions. The only other intersection is near $\nu \approx 1.54$. In general, we cannot find closed forms in solving these sorts of equations where transcendental functions are mixed with rational functions. In applications, these might be solved numerically using Newton’s method, or, perhaps, if the high accuracy is not needed, read off of a graph as above. If we name the root of this equation $\nu_0 \approx 1.54$, then we can find the corresponding values of $A$ and of $B$. When we row reduce the matrix, it must end up with a zero second row, and so the solution $\begin{pmatrix} A \\ B \end{pmatrix}$ will be proportional to $\begin{pmatrix} \nu_0 - 1 \\ \nu_0 + 1 \end{pmatrix}$ leading to an eigenfunction of $y(x) = (\nu_0 - 1)e^{\nu_0 x} + (\nu_0 + 1)e^{-\nu_0 x}$, with $\lambda = -\nu_0^2$, and $\nu_0$ defined as the unique positive root of $e^\nu = \left| \frac{1 + \nu}{1 - \nu} \right|$. 

The remaining case is that of positive $\lambda$. For convenience let $\lambda = k^2$ with $k > 0$. Then $y = A\cos(kx) + B\sin(kx)$ and the boundary condition lead to:\n$y(0) + y'(0) = A + kB = 0$, and $y(0) - y'(0) = (A\cos(k) + B\sin(k)) - (-k\sin(k) + k\cos(k)) = 0$.
In matrix form this is \[
\begin{pmatrix}
1 \\
\cos(k) + k\sin(k)
\end{pmatrix}
\begin{pmatrix} A \\ B \end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Setting the determinant equal to zero gives

\[(1 - k^2) \sin(k) - 2k \cos(k) = 0.\]

It is not hard to check that when \(\sin(k) = 0\), so that \(k = n\pi > 0\) this is \(-2n\pi \cos(n\pi) \neq 0\). Thus we can divide through by \(\sin(k)\) to get

\[\cot(k) = \frac{1 - k^2}{2k}.\]

Once again, we have an equation mixing a transcendental function (the cotangent in blue) and a rational function \(((1 - k^2)/2k\) in red).

Looking at the graph, we see that there are an infinite number of places where the two curves intersect. As many of these as we desire could be solved for numerically. For example, \(k_1 \approx 2.331\). If we denote the roots by \(k_n\), with \(n = 1, 2, \ldots\), then the eigenvalues are \(\lambda = k_n^2\), and the eigenfunctions can be found by using the first row of the matrix, \((1, k_n)\), so that \(\begin{pmatrix} A \\ B \end{pmatrix}\) will be proportional to \(\begin{pmatrix} k_n \\ -1 \end{pmatrix}\), and the eigenfunctions are \(y_n(x) = k_n \cos(k_n x) - \sin(k_n x)\).