Section 5.2: 15 Suppose that you have a system of differential equations $dx/dt = Ax$, where $x(t)$ is a function taking values in $\mathbb{R}^n$, and $A \in M_{n \times n}(\mathbb{R})$. Which is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, not necessarily distinct. We can find a basis $\beta$ of $n$ eigenvectors $v_i$. Since $\beta$ is a basis, we can write $x(t) = \sum_{i=1}^n a_i(t)v_i$, where the coordinate functions $a_i(t)$ are uniquely determined by $x(t)$. Substituting this expansion in the system of differential equations gives

$$\sum_{i=1}^n a_i'(t)v_i = \sum_{i=1}^n a_i(t)Av_i = \sum_{i=1}^n a_i(t)\lambda_i v_i.$$  

The coordinates with respect to the basis are unique, so we can set the coefficients of the $v_i$ equal to each other to get

$$a_i'(t)v_i = \lambda_i a_i(t).$$

which implies that $a_i(t) = a_i(0)e^{\lambda_it}$. Thus we have

$$x(t) = \sum_{i=1}^n a_i(0)e^{\lambda_it}v_i.$$  

Combining terms with a common value for $\lambda_i$ yields

$$x(t) = \sum_{\text{distinct } \lambda} e^{\lambda t}z_\lambda,$$

where the $z_\lambda = \sum_i$ with $\lambda_i = \lambda$ $a_i(0)v_i$ is a linear combination of eigenvectors with eigenvalues $\lambda$, and thus is in $E_\lambda$.

On the other hand it is straightforward to check that any $x(t)$ of this form is a solution, thus the space of solutions is spanned by the set $e^{\lambda_it}v_i$, which is a linear independent set. Thus the set of solutions is a vector space of dimension $n$.

Section 5.2: 20 and 21 Let $W_i$, $i = 1, \ldots, k$ be subspaces of a finite dimensional vector space $V$ such that $\sum_{i=1}^n W_i = V$. If $V$ is the direct sum of the $W$’s, then $W_j \cap \sum_{i \neq j} W_i = \{0\}$. Suppose that dim($W_i$) = $n_i$ and that $\beta_i = \{v_{i1}, \ldots, v_{in_i}\}$ is a basis of $W_i$. If we have coefficients $a_{ij}$, so that $\sum_{i=1}^k \sum_{j=1}^{n_i} a_{ij}v_{ij} = 0$, then we can then write

$$\sum_{j=1}^{n_k} a_{kj}v_{kj} = -\sum_{i=1}^{k-1} \sum_{j=1}^{n_i} a_{ij}v_{ij} \in W_k \cap \sum_{i=1}^{k-1} W_i = \{0\}.$$  

Since the $v_{kj}$ are a basis of $W_k$ and thus linearly independent, the $a_{kj} = 0$. This argument can be used inductively to show that all of the coefficients must be zero, and thus the union of the $\beta_i$ is a linearly independent set. It also spans $V$, and thus it is a basis. It has $\sum n_i$ elements, and so dim($V$) = $\sum$ dim($W_i$).

Conversely, suppose that dim($V$) = $\sum$ dim($W_i$). The vectors $v_{ij}$ span $\sum W_i$. If there is a nontrivial linear dependence relation among these vectors, then there is a spanning set for $V$ that has fewer than $\sum n_i$ elements. This is not possible since any spanning set must have at least dim($V$) elements. Thus The $v_{ij}$ are linearly independent. Suppose that $u \in W_m \cap \sum_{i \neq m} W_i$. Then we can write $u = \sum_{j=1}^{n_m} a_{mj}v_{mj} = \sum_{i \neq m} \sum_{j=1}^{n_i} a_{ij}v_{ij}$. This leads to

$$u = \sum_{i \neq m} \sum_{j=1}^{n_i} a_{ij}v_{ij}.$$  

Since the $v_{ij}$ are a basis of $W_k$ and thus linearly independent, the $a_{ij} = 0$. This argument can be used inductively to show that all of the coefficients must be zero, and thus the union of the $\beta_i$ is a linearly independent set. It also spans $V$, and thus it is a basis. It has $\sum n_i$ elements, and so dim($V$) = $\sum$ dim($W_i$).
a dependency relationship among the $v_{ij}$ which implies that $a_{ij} = 0$ and so $u = 0$. Thus we
have shown that $V$ is a direct sum of the $W_i$.

Problem 21 follows immediately by setting $W_i = \text{span}\{\beta_i\}$.

**Section 5.2: 22** Let $T$ be a linear operator on a finite dimensional vector space $V$ with
distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Let

$$U = \text{span}\{x \in V : x \text{ is an eigenvector of } T\} = \text{span}\{x \in V : x \in \bigcup_{i=1}^k E_{\lambda_i} x \neq 0\}.$$ 

By its definition, $U = \sum E_{\lambda_i}$. Suppose that $u \in E_{\lambda_m} \cap \sum_{i \neq m} E_{\lambda_i}$, then we can write

$$u = w_m = -\sum_{i \neq m} w_i$$

with $w_j \in E_{\lambda_j}$, and rearranging we have $\sum_i w_i = 0$. As in the lemma on page 267, this
implies that all of the $w_j = 0$, and thus that $u = 0$. Thus the intersection $E_{\lambda_m} \cap \sum_{i \neq m} E_{\lambda_i} = 0$,
and $U$ is a direct sum of the $E_{\lambda_i}$. 