Algebraic geometry and stability for integrable systems

Anton Izosimov

University of Toronto, Canada
Moscow State University, Russian Federation

HIGHLIGHTS

- We present an algebro-geometric method of stability analysis for integrable systems.
- The method allows to conclude about stability from the shape of the spectral curve.
- We solve the stability problem for the free n-dimensional rigid body as an example.

ABSTRACT

In 1970s, a method was developed for integration of nonlinear equations by means of algebraic geometry. Starting from a Lax representation with spectral parameter, the algebro-geometric method allows to solve the system explicitly in terms of theta functions of Riemann surfaces. However, the explicit formulas obtained in this way fail to answer qualitative questions such as whether a given singular solution is stable or not. In the present paper, the problem of stability for equilibrium points is considered, and it is shown that this problem can also be approached by means of algebraic geometry.

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1. Introduction

As is well-known, many finite-dimensional integrable systems can be explicitly solved by means of algebraic geometry. The starting point for the algebro-geometric integration method is Lax representation. A dynamical system is said to admit a *Lax representation with spectral parameter* λ if the following two conditions are satisfied.

1. The phase space of the system can be identified with a certain submanifold $\mathcal{L}$ of the space $\mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}(\lambda)$ of matrix-valued functions of a complex variable λ.
2. Under this identification, equations of motion take the form

$$\frac{\mathrm{d}}{\mathrm{d}t} L_\lambda = [L_\lambda, A_\lambda(L_\lambda)] \qquad (1)$$

where $L_\lambda \in \mathcal{L}$ is the phase variable, and A_λ is a mapping $A_\lambda \colon \mathcal{L} \to \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}(\lambda)$.

Starting from a Lax representation with spectral parameter, the algebro-geometric integration method allows to write the solution of the system with initial data $L_\lambda = L_\lambda^0$ in terms of theta functions associated with the algebraic curve

$$\det(L_\lambda^0 - \mu \mathrm{E}) = 0, \qquad (2)$$

which is called the *spectral curve*. We refer the reader to [1–5] and references therein for more details on the algebro-geometric integration method.

Despite the possibility to explicitly solve Eq. (1) in terms of theta functions, if we are interested in qualitative features of dynamics, theta-functional formulas seem to be of little use at least for the following reasons. Firstly, theta-functional solutions correspond to non-singular spectral curves, while most remarkable solutions, such as fixed points or stable periodic solutions, are related to degenerate curves. Secondly, theta-functional formulas provide solutions of the complexified system, and it is in general a difficult problem to describe real solutions. At the same time, many dynamical phenomena, such as stability, are related to the presence of a real structure.

In the present paper we study the Lyapunov stability problem for systems which admit a Lax representation with spectral parameter. We show that this problem can also be approached by means

E-mail address: a.m.izosimov@gmail.com.

of classical algebraic geometry, and that this approach is very natural and fruitful. Though we focus on stability of equilibrium points, we expect that our results can be generalized to more general solutions. We note that the relation between topology of integrable systems and algebraic geometry has been extensively studied by M. Audin and her collaborators [5,6], however it seems that their approach cannot be directly used to study the stability problem.

Before we formulate the main result of the paper, let us describe the class of Lax matrices which we consider. Firstly, for the sake of simplicity, we restrict ourselves to the case when L_λ is polynomial in parameter λ, i.e. when $\mathcal{L} \subset \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda]$. Note that it is more standard to consider Lax matrices which are polynomial in λ and λ^{-1}, i.e. which belong to the loop algebra $\mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. However this situation can be reduced to the polynomial case by multiplying L_λ by a suitable power of λ.

Our second assumption is the following: there exists an anti-holomorphic involution $\tau: \mathbb{C} \to \mathbb{C}$ and a complex number σ of absolute value 1 such that for each $L_\lambda \in \mathcal{L}$

$$L_{\tau(\lambda)} = \sigma L_\lambda^*. \tag{3}$$

The presence of such involution is a common feature of many integrable systems. The most standard examples are provided in Table 1. See [7–11] for more details on these systems and their Lax representations.

Let us now formulate the main result. Let $L_\lambda^0 \in \mathcal{L}$, and consider the associated spectral curve (2). The involution τ induces an anti-holomorphic involution

$$\widehat{\tau}: (\lambda, \mu) \mapsto (\tau(\lambda), \sigma\bar{\mu})$$

on the spectral curve. We show that if L_λ^0 is a fixed point of the Lax equation (1), then, under some additional assumptions, a sufficient condition for its stability is that all singular points of the associated spectral curve are fixed points of $\widehat{\tau}$. If we interpret $\widehat{\tau}$ as a real structure, then this condition means that all singular points lie in the real part of the curve. We consider several examples in which this condition turns out to be necessary and sufficient.

Our first example is the Lagrange top. We use the algebro-geometric approach to recover the classical result that the rotation of a top is stable if the angular velocity is greater or equal than some critical value.

The second example is the top on a compact Lie algebras defined by a *sectional operator*. This system is related to the so-called *argument shift method*, see Mishchenko and Fomenko [12].

The third, and the most interesting, example is the free multidimensional rigid body, or Euler–Manakov top. It is a standard result that the rotation of a torque-free three-dimensional rigid body about the short or the long axis of inertia is stable, whereas the rotation about the middle axis is unstable. Using the algebro-geometric approach, we obtain a multidimensional generalization of this result. We note that this problem has previously been approached by different methods [13–18], however no complete solution has been known.

2. Stability for integrable and Lax systems

Let $\dot{x} = v(x)$ be a dynamical system on a manifold X, and assume that $f_1, \ldots, f_N$ are its (in general, complex-valued) first integrals. The *moment map* is a map $F: X \to \mathbb{C}^N$ which maps $x \in X$ to $(f_1(x), \ldots, f_N(x))$.

Proposition 2.1. *Assume that $x_0 \in X$ is an isolated point in the level set of the moment map. Then x_0 is Lyapunov stable fixed point of $\dot{x} = v(x)$.*

Table 1
Integrable systems which admit Lax representation with $L_{\tau(\lambda)} = \sigma L_\lambda^*$.

Integrable system	Lax matrix	τ	σ
Euler–Manakov top	$L_\lambda = A + B\lambda,$ $A^* = -A, B^* = B$	$\lambda \mapsto -\bar{\lambda}$	-1
Kowalewski top	$L_\lambda = A + B\lambda + C\lambda^2,$ $A^* = A, B^* = -B, C^* = C$	$\lambda \mapsto -\bar{\lambda}$	1
Geodesic flow on ellipsoid	$L_\lambda = A + B\lambda + C\lambda^2,$ $A^* = A, B^* = -B, C^* = C$	$\lambda \mapsto -\bar{\lambda}$	1
Lagrange top	$L_\lambda = A + B\lambda + C\lambda^2,$ $A^* = -A, B^* = -B, C^* = -C$	$\lambda \mapsto \bar{\lambda}$	-1

Proof. Let $x(t)$ be the solution with $x(0) = x_0$. Then $F(x(t)) = F(x(0))$, so $x(t) \in F^{-1}(F(x_0))$. Since x_0 is isolated in $F^{-1}(F(x_0))$, this implies that $x(t) = x_0$, i.e. x_0 is a fixed point. To prove stability, note that

$$f(x) = \sum_{i=1}^{N} |f_i(x) - f_i(x_0)|^2$$

is a Lyapunov function. $\square$

Remark 2.1. As was shown by Bolsinov and Borisov [19], a similar statement is true for periodic trajectories: if a periodic trajectory coincides with a connected component of the level set of the moment map, then it is stable. Moreover, under some additional assumptions, the converse is also true. In our case, the following is true. Let X be Poisson manifold, and let $\dot{x} = v(x)$ be a Hamiltonian system. Assume that $f_1, \ldots, f_N$ is a complete family of analytic first integrals in involution. Further, assume that the level sets of F are compact, so that their connected components are invariant tori, and that $\dot{x} = v(x)$ is a non-resonant system, which means that its trajectories are dense on almost all tori, see Bolsinov and Fomenko [20]. Then the condition of Proposition 2.1 is necessary and sufficient.

Now, let us reformulate Proposition 2.1 for systems which admit a Lax representation with spectral parameter. Consider the space

$$\mathcal{P}_{m,n} = \{L_\lambda = B_m\lambda^m + \cdots + B_0 \in \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda]\}$$

of $\mathfrak{gl}(n, \mathbb{C})$-valued polynomials of degree m, and let $\mathcal{L} \subset \mathcal{P}_{m,n}$ be its submanifold. Let A_λ be a map $A_\lambda: \mathcal{L} \to \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}(\lambda)$, and assume that $\mathcal{L}$ is invariant with respect to the flow

$$\frac{d}{dt}L_\lambda = [L_\lambda, A_\lambda(L_\lambda)]. \tag{4}$$

To each $L_\lambda \in \mathcal{L}$ we can assign its spectral curve, i.e. an affine algebraic curve $C(L_\lambda)$ given by the equation $P(\lambda, \mu) = 0$ where

$$P(\lambda, \mu) = \det(L_\lambda - \mu E)$$

is the characteristic polynomial of L_λ. The following is well-known.

Proposition 2.2. *Let $L_\lambda(t)$ be a solution of (4). Then the curve $C(L_\lambda(t))$ does not depend on t.*

Proof. Eq. (4) implies that

$$\frac{d}{dt}(L_\lambda)^k = [(L_\lambda)^k, A_\lambda]$$

for any positive integer k. Therefore, the function $\mathrm{Tr}\,(L_\lambda)^k$ is an integral of motion for any values of k and λ, and so are the coefficients of the characteristic polynomial $P(\lambda, \mu)$. $\square$

Proposition 2.3. *Let $L_\lambda^0 \in \mathcal{L}$, and assume that L_λ^0 is an isolated point in the isospectral variety*

$$S(L_\lambda^0) = \{L_\lambda \in \mathcal{L} \mid C(L_\lambda) = C(L_\lambda^0)\}.$$

Then L_λ^0 is a stable fixed point of (4).

Proof. This follows from Propositions 2.1 and 2.2. □

In the next section, we present algebro-geometric conditions which imply the hypothesis of Proposition 2.3.

3. A lemma on polynomial matrix pencils and the stability theorem

Let $B \in \mathfrak{gl}(n, \mathbb{C})$ be a fixed $n \times n$ matrix, and let

$$\mathscr{P}_{m,n}(B) = \{L_\lambda = B_m \lambda^m + \cdots + B_0 \in \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda] \mid B_m = B\}$$

be the set of $\mathfrak{gl}(n, \mathbb{C})$-valued polynomials of degree m with leading coefficient B. To each matrix polynomial $L_\lambda \in \mathscr{P}_{m,n}(B)$ we assign its spectral curve, i.e. an affine algebraic curve $C(L_\lambda)$ given by the equation $P(\lambda, \mu) = 0$ where

$$P(\lambda, \mu) = \det(L_\lambda - \mu E).$$

The space $\mathscr{P}_{m,n}(B)$ carries an action of the gauge group

$$G = \{Q \in \mathrm{GL}(n, \mathbb{C}) \mid [Q, B] = 0\} / \{\nu E \mid \nu \in \mathbb{C}^*\}.$$

This action is given by $L_\lambda \mapsto Q^{-1} L_\lambda Q$, and for each polynomial $L_\lambda \in \mathscr{P}_m(B)$ we have

$$C(L_\lambda) = C(Q^{-1} L_\lambda Q),$$

i.e. the spectral curve is the same for matrix polynomials belonging to the same gauge group orbit. However, the converse is not true: two matrix polynomials which have the same spectral curve do not necessarily belong to the same gauge group orbit.

Remark 3.1. More precisely, let us consider the variety of matrix polynomials isospectral with L_λ, i.e. the set

$$S(L_\lambda) = \{M_\lambda \in \mathscr{P}_{m,n}(B) \mid C(M_\lambda) = C(L_\lambda)\}.$$

Then, if we assume that the spectral curve of L_λ is non-singular, and that B has simple spectrum, the quotient $S(L_\lambda)/G$ can be identified with a Zariski open subset of the Jacobian of the spectral curve, see van Moerbeke and Mumford [21]. In the case when the spectral curve is singular, the description of the quotient is not that transparent, however it is true that $S(L_\lambda)/G$ contains a Zariski open subset which can be identified with a Zariski open subset of the generalized Jacobian of the spectral curve [21]. In particular, the dimension of the isospectral variety is still much bigger than the dimension of the G-orbit.

The situation changes drastically if we restrict our attention to matrix-valued polynomials which admit an anti-holomorphic involution. Let $\tau \colon \mathbb{C} \to \mathbb{C}$ be an anti-holomorphic involution, and let $\sigma \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Define

$$\widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma) = \{L_\lambda = B_m \lambda^m + \cdots + B_0 \in \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[\lambda] \mid$$
$$L_{\tau(\lambda)} = \sigma L_\lambda^*, \ B_m = B\}.$$

In this case, the gauge group is

$$G = \{Q \in \mathrm{U}(n) \mid [Q, B] = 0\} / \{\nu E \mid \nu \in S^1\}.$$

For each $L_\lambda \in \widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma)$, there is an anti-holomorphic involution

$$\widehat{\tau} \colon (\lambda, \mu) \mapsto (\tau(\lambda), \sigma \overline{\mu})$$

on its spectral curve. Denote by $\mathrm{Fix}\,\widehat{\tau}$ the fixed points set of this involution. If we interpret $\widehat{\tau}$ as a real structure, then $\mathrm{Fix}\,\widehat{\tau}$ is the set of real points on the curve.

Lemma 1. Let $L_\lambda^0 \in \widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma)$, and assume that the leading term B has simple spectrum. Let $C = C(L_\lambda^0)$ be the spectral curve associated with L_λ^0. Assume that

1. all singular points of C lie in $\mathrm{Fix}\,\widehat{\tau}$;
2. each irreducible component of C is smooth and has genus zero;
3. each intersection between two distinct irreducible components is at worst of order 2.

Then the isospectral variety

$$S = \{L_\lambda \in \mathscr{P}_{m,n}(B) \mid C(L_\lambda) = C\}$$

coincides with the gauge group orbit of L_λ^0, i.e. if $L_\lambda \in \widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma)$, and $C(L_\lambda) = C$, then there exists $Q \in G$ such that

$$L_\lambda = Q^{-1} L_\lambda^0 Q.$$

The proof is given in Section 8.

Remark 3.2. We note that since each irreducible component of C is smooth, the singularities of C are exactly those points which belong to at least two components. At the same time, the smoothness condition can actually be avoided, however if irreducible components of C have self-intersections, they should also satisfy Conditions 1 and 3 of the lemma. We require smoothness not to go deep into singularity theory of algebraic curves. In all examples that we consider, irreducible components of the spectral curve are indeed smooth.

Condition 3 can also apparently be weakened. However, in all our examples, irreducible components of the spectral curve are either lines or quadrics, so this condition is automatically satisfied.

The two remaining conditions, namely Condition 1 and condition on the genus, are crucial. In Section 7 we consider several counterexamples which show that Lemma 1 does not in general hold if one of these two conditions is not satisfied. However, Lemma 1 admits the following generalization on higher genus curves: if all conditions of Lemma 1 are satisfied except, possibly, the condition on the genus, then the quotient of the isospectral variety by the action of the gauge group can be identified with a subset of the Jacobian of C. Note that, in the genus zero case, this is exactly Lemma 1, since the Jacobian is a single point. Also note that if Condition 1 of Lemma 1 is violated, then the Jacobian should be replaced by the generalized Jacobian.

Now we are in a position to state the main result of the paper.

Theorem 1. Consider a system which admits a Lax representation with spectral parameter such that its phase space $\mathscr{L}$ lies in $\widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma)$. Assume that

1. $L_\lambda^0 \in \mathscr{L}$ satisfies conditions of Lemma 1;
2. the intersection of the gauge group orbit of L_λ^0 with $\mathscr{L}$ is a discrete set.

Then L_λ^0 is a Lyapunov stable fixed point.

Proof. Lemma 1 implies that the isospectral variety

$$S(L_\lambda^0) = \{L_\lambda \in \widetilde{\mathscr{P}}_{m,n}(B, \tau, \sigma) \mid C(L_\lambda) = C(L_\lambda^0)\}$$

coincides with G-orbit of L_λ^0, so

$$\{L_\lambda \in \mathscr{L} \mid C(L_\lambda) = C(L_\lambda^0)\} = S(L_\lambda^0) \cap \mathscr{L}$$

is a discrete subset of $\mathscr{L}$. Now apply Proposition 2.3. □

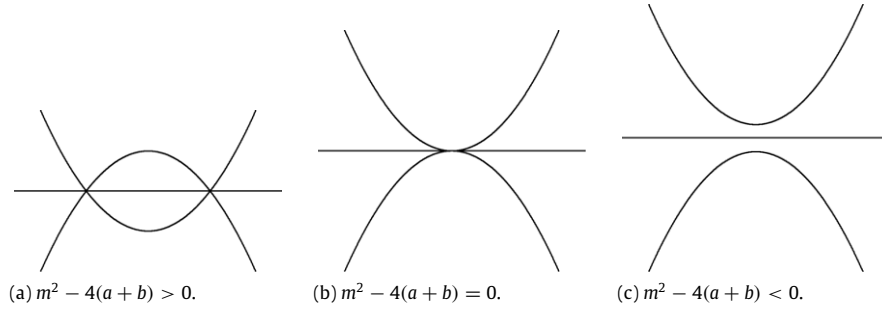(a) $m^2 - 4(a+b) > 0$.   (b) $m^2 - 4(a+b) = 0$.   (c) $m^2 - 4(a+b) < 0$.

Fig. 1. Spectral curves for the sleeping top.

Remark 3.3. Note that the Lax flow (4) can be, as a rule, included into an hierarchy of commuting Lax flows

$$\frac{d}{dt}L_\lambda = [L_\lambda, A^i_\lambda(L_\lambda)], \quad i = 1, \ldots, k. \tag{5}$$

Theorem 1 is applicable for all these flows. In particular, if $L^0_\lambda \in \mathcal{L}$ satisfies the assumptions of the theorem, then it is a common equilibrium point for all flows (5). Note that if $L^0_\lambda \in \mathcal{L}$ is an equilibrium point of (4), but is not a common equilibrium point, then L^0_λ is automatically not isolated in the moment map level set, and under some additional assumptions, it is unstable (see Remark 2.1).

In Sections 4–6 we consider three examples illustrating Theorem 1. In Section 7 we consider several counterexamples showing that assumptions of Lemma 1 or Theorem 1 in general cannot be avoided (except those mentioned in Remark 3.2).

4. Example I: Lagrange top

The equations of the Lagrange top read

$$\begin{cases} \dot{M} = [M, \Omega] + [\Gamma, \chi], \\ \dot{\Gamma} = [\Gamma, \Omega], \end{cases}$$

where $M, \Gamma \in \mathfrak{so}(3, \mathbb{R})$ are dynamical variables, $\Omega \in \mathfrak{so}(3, \mathbb{R})$ is defined by the relation $M = \Omega J + J\Omega$,

$$\chi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(a, b, b)$$

where $a, b \in \mathbb{R}$ are the inertia moments of the top. These equations can be rewritten as a single Lax equation with parameter:

$$\frac{d}{dt}(\lambda^2(a+b)\chi + \lambda M + \Gamma) = [\lambda^2(a+b)\chi + \lambda M + \Gamma, \lambda\chi + \Omega].$$

See [10,22,23] for a detailed discussion of this system and underlying algebraic geometry.

Note that the Lax matrix $L_\lambda = \lambda^2(a+b)\chi + \lambda M + \Gamma$ satisfies $L_{\tau(\lambda)} = -L^*_\lambda$ with τ given by $\lambda \mapsto \bar{\lambda}$, which allows the application of Theorem 1. Consider the fixed point (M_0, Γ_0) where $\Gamma_0 = \chi$ and $M_0 = m\chi$. From the point of view of mechanics, this point corresponds to a rotation about the vertical axis, or the so-called sleeping top. It is a classical result that the sleeping top with $m^2 \geq 4(a+b)$ is stable. Let us recover this result by means of the spectral curve. The equation of the spectral curve associated with $L^0_\lambda = \lambda^2(a+b)\chi + \lambda M_0 + \Gamma_0$ is

$$P(\lambda, \mu) = -\mu(\mu + i((a+b)\lambda^2 + m\lambda + 1)^2) \\ \times (\mu - i((a+b)\lambda^2 + m\lambda + 1)^2) = 0.$$

This curve consists of three irreducible components C_1, C_2, C_3 of genus 0. Since the involution $\hat{\tau}$ is $(\lambda, \mu) \mapsto (\bar{\lambda}, -\bar{\mu})$, we sketch the

spectral curve in the axes $\lambda, \mu i$. Three sketches in Fig. 1 correspond to the cases $D = m^2 - 4(a+b) > 0, D = 0$, and $D < 0$. We conclude that the conditions of Lemma 1 hold if and only if $D \geq 0$, i.e. if the rotation is sufficiently fast.

Now, to prove that the sleeping top with $D \geq 0$ is stable, it suffices to find the intersection of the gauge group orbit with the phase space. Let $Q \in G$, and let

$$L_\lambda = \lambda^2(a+b)\chi + \lambda M + \Gamma = Q(\lambda^2(a+b)\chi + \lambda M_0 + \Gamma_0)Q^{-1}.$$

We have $Q\chi Q^{-1} = \chi$, so $M = QM_0Q^{-1} = mQ\chi Q^{-1} = m\chi = M_0$. Analogously, $\Gamma = \Gamma_0$, so the G-orbit of L^0_λ is one point, and the equilibrium (M_0, Γ_0) is stable.

The condition $D \geq 0$ is actually necessary and sufficient for stability, see e.g. Arnold [24]. The case when $D = 0$ is more complicated compared to $D > 0$. It corresponds to the so-called *supercritical Hamiltonian Hopf bifurcation*, see Cushman and Meer [25].

5. Example II: Mishchenko–Fomenko tops on compact Lie algebras

Let $\mathfrak{g}$ be a compact simple Lie algebra. The equations of the Mishchenko–Fomenko top [12] are

$$\dot{X} = [X, \phi(X)] \tag{6}$$

where $X \in \mathfrak{g}$, and $\phi \colon \mathfrak{g} \to \mathfrak{g}$ is a linear operator satisfying

$$[\phi(X), A] = [X, B] \tag{7}$$

for some regular $A \in \mathfrak{g}$, and $B \in \mathfrak{C}(a)$ where $\mathfrak{C}(a) = \{X \in \mathfrak{g} \mid [X, A] = 0\}$ is the centralizer of A. Operators satisfying (7) are called *sectional operators*. These operators were introduced in Fomenko and Trofimov [26], see also Bolsinov and Konyaev [27] and references therein.

The Lax representation with parameter for the system (6) reads

$$\frac{d}{dt}(X + \lambda A) = [X + \lambda A, \phi(X) - \lambda B].$$

Note that the Lax matrix $L_\lambda = X + \lambda A$ is a $\mathfrak{g}$-valued polynomial. To obtain a matrix-valued polynomial, we pass to any unitary representation $\rho \colon \mathfrak{g} \to \mathfrak{u}(n)$. The Lax matrix so obtained satisfies $L_{\tau(\lambda)} = -L^*_\lambda$ with τ given by $\lambda \mapsto \bar{\lambda}$.

Proposition 5.1. Let $X_0 \in \mathfrak{C}(A)$. Then X_0 is a fixed point of (6).

Proof. Let $X_0 \in \mathfrak{C}(A)$. Since A is regular, $\mathfrak{C}(A)$ is Abelian, and since $B \in \mathfrak{C}(A)$, we have $[X_0, B] = 0$. Using (7), we conclude that $\phi(X_0) \in \mathfrak{C}(A)$, so $[X_0, \phi(X_0)] = 0$, q.e.d. □

Let us use Theorem 1 to prove that all these equilibria are stable. Note that though A is regular, the spectrum of its image under the representation ρ may be not simple. For example, a matrix $A \in \mathfrak{so}(2n, \mathbb{R})$ with two zero eigenvalues is a regular element. However, it is easy to see that if $\mathfrak{g}$ is of type A_n, B_n, C_n, or G_2, and ρ is the representation of minimal dimension, then regularity of A

does imply simplicity of the spectrum of $\rho(A)$. See Konyaev [28] for details.

So, let A and X_0 be represented by $n \times n$ matrices. Since $X_0 \in \mathfrak{C}(A)$, the matrices X_0 and A are simultaneously diagonalizable. The equation of the spectral curve is

$$P(\lambda, \mu) = \prod_{i=1}^{n} (x_i + \lambda a_i - \mu) = 0$$

where $\{a_i\} \subset i\mathbb{R}$ and $\{x_i\} \subset i\mathbb{R}$ are eigenvalues of A and X_0 respectively. So, the spectral curve is the union of distinct straight lines $\mu = a_i \lambda + x_i$. It is clear that all conditions of Lemma 1 are satisfied, and the gauge group orbit of $L_\lambda^0 = X_0 + \lambda A$ consists of one point. Therefore, X_0 is stable.

Note that it is possible to generalize Lemma 1 in such a way that D_n, E_6, E_7, E_8, and F_4 can also be included. The stability conclusion remains true for these Lie algebras as well. This can also be proved using the technique of [17].

Remark 5.1. We note that the Eq. (6) is completely integrable can be included into an hierarchy of commuting flows. Equilibrium points described in Proposition 5.1 are exactly those which are common for all these flows, see Brailov [29] and Bolsinov and Oshemkov [30]. The Eq. (6) may have other equilibria, however their stability cannot be studied by the method of the present paper (see Remark 3.3). Nevertheless, if we are able to prove that (6) is a non-resonant system, then we can assert that these equilibria are unstable (see Remark 3.3). Non-resonance of (6) for generic B can be deduced from the results of Rybnikov [31].

6. Example III: Multidimensional rigid body

The equations of motion of a torque-free multidimensional rigid body, also known as the Euler–Manakov top, are

$$\dot{M} = [M, \Omega], \tag{8}$$

$M \in \mathfrak{so}(n, \mathbb{R})$ is a dynamical variable, and Ω is found from the equation $M = J\Omega + \Omega J$ where J is a fixed symmetric matrix.

This example is the most interesting from the point of view of stability. A complete solution of the stability problem is only known for the three-dimensional body. In three dimensions, the equilibria of (8) are rotations about principal axes, and it is well known that the rotation about the short and the long axis is stable, while the rotation about the middle axis is unstable. For a detailed discussion of the stability problem for the multidimensional rigid body, see [18]. In this section we show that the solution of this problem by means of the spectral curve is almost straightforward, at least for the so-called *regular* equilibria which are defined below (see also Remark 6.3).

Let M_0 be an equilibrium point of (8). Then M_0 is called regular [32] if it is possible to bring J and M_0 to the canonical form simultaneously, i.e. if there exists a basis where J is diagonal and M_0 takes the form

$$M_0 = \begin{pmatrix} 0 & m_1 & & & & & \\ -m_1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & m_l & & \\ & & & -m_l & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

Regular equilibria are a natural multidimensional generalization of rotations about principal axes. A complete solution of the stability problem for regular equilibria under the condition that J is generic is given below.

A Lax representation with parameter for the system (8) was found by Manakov [7]. It reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}(M + \lambda J^2) = [M + \lambda J^2, \Omega + \lambda J]. \tag{9}$$

The involution τ is $\lambda \mapsto -\bar{\lambda}$.

Let M_0 be a regular equilibrium point, and let $J^2 = \mathrm{diag}(a_1, \ldots, a_n)$ where the numbers $a_1, \ldots, a_n$ are all distinct. Then the spectral curve is given by

$$\begin{aligned} P(\lambda, \mu) &= \prod_{i=1}^{l} \left(m_i^2 + (a_{2i-1}\lambda - \mu)(a_{2i}\lambda - \mu)\right) \prod_{i=2l+1}^{n} (a_i\lambda - \mu) \\ &= 0, \end{aligned}$$

thus it is a union of hyperbolas and straight lines. To investigate when the conditions of Lemma 1 are satisfied, we make a change of variables

$$\begin{cases} x = \dfrac{\mu}{\lambda}, \\ y = -\dfrac{1}{\lambda^2} \end{cases}$$

and obtain the curves

$$y = \frac{(x - a_{2i-1})(x - a_{2i})}{m_i^2}, \quad i = 1, \ldots, l.$$
$$x = a_i, \quad i = 2l + 1, \ldots, n.$$

The union of these curves is called the *parabolic diagram* associated with M_0. It is obvious that the only condition of Lemma 1 which needs to be checked is that all singular points of the spectral curve belong to the set Fix $\hat{\tau}$ where $\hat{\tau}$ is given by $(\lambda, \mu) \rightarrow (-\bar{\lambda}, -\bar{\mu})$. In terms of the parabolic diagram, this condition means that all intersections are real and belong to the set $\{x \in \mathbb{R}, y > 0\} \cup \{y = \infty\}$. To show that this condition implies stability, we need to describe the gauge group orbit of $L_\lambda^0 = M_0 + \lambda J^2$. The gauge group consists of diagonal unitary matrices, so the G-orbit of M_0 in this case is not discrete. However, its intersection with the phase space, i.e. with the set of real skew-symmetric matrices, is finite and consists of matrices

$$\begin{pmatrix} 0 & \pm m_1 & & & & & \\ \mp m_1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & \pm m_l & & \\ & & & \mp m_l & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

We conclude that if all intersections on the parabolic diagram are either real and belong to the upper half-plane, or infinite, then the equilibrium is stable. On the other hand, it is shown in [18] that this condition is also a necessary condition for stability, which implies the following:

Theorem 2. *A regular equilibrium of the torque-free multidimensional rigid body is stable if and only if all singular points on the associated parabolic diagram are either real and belong to the upper half-plane, or infinite.*

A weaker version of this result was proved in [18] by means of the bi-Hamiltonian approach. It included an additional requirement that there are no tangency points on the parabolic diagram (i.e. all intersections are of order 1).

Parabolic diagrams for the three-dimensional body are depicted in Fig. 2. Classical results on stability are immediately recovered. Much more examples of parabolic diagrams can be found in [18].
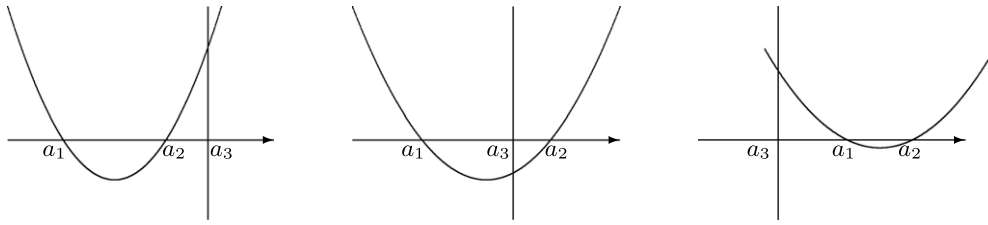
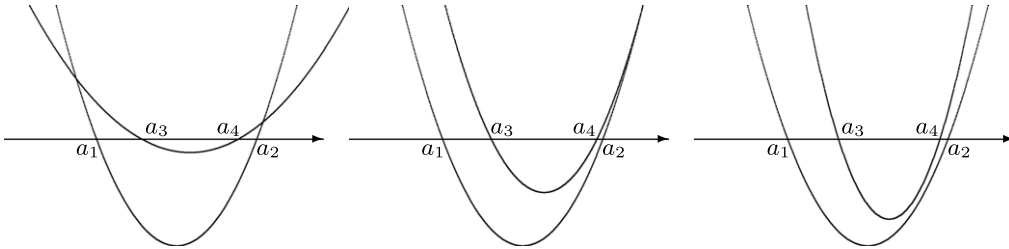**Fig. 2.** Parabolic diagrams for the three-dimensional body.

Fig. 3. Stability loss under Hamiltonian Hopf bifurcation for the four-dimensional rigid body.

Remark 6.1. Parabolic diagrams were introduced in [17] to describe the spectrum of the bi-Hamiltonian structure related to the Eq. (8). In the context of spectral curves, the parabolic diagram is the quotient of the spectral curve by the holomorphic involution $(\lambda, \mu) \rightarrow (-\lambda, -\mu)$. The relation between the bi-Hamiltonian structure and the quotient of the spectral curve remains unclear.

Remark 6.2. The stability problem for the four-dimensional rigid body was solved almost completely in [13,16]. However, there is one case in which the instability conclusions of these papers seem to be incorrect. This is case IIIb of (3.13) in Fehér and Marshall [13], and case V in Theorem 5.3 of Birtea et al. [16]. It is the case with a tangency point in the upper half-plane depicted in the second diagram in Fig. 3. Theorem 2 implies that this equilibrium is stable. This point lies on the boundary of the set of stable equilibria and corresponds to the Hamiltonian Hopf bifurcation (see Section 4). As the ratio $m_1^2 : m_2^2$ grows, a stable regime (first diagram in Fig. 3) is being replaced by an unstable regime (third diagram in Fig. 3). Note that this bifurcation is only possible if $[a_1, a_2] \supset [a_3, a_4]$, or $[a_1, a_2] \subset [a_3, a_4]$. If $[a_1, a_2] \cap [a_3, a_4]$ is empty, then the rotation is stable. And if it is a proper subset of both $[a_1, a_2]$ and $[a_3, a_4]$, then the rotation is unstable.

Remark 6.3. There also exist equilibria of the Eq. (8) which do not satisfy the regularity condition—the so-called *exotic equilibria* [32]. The difference between regular and exotic equilibria is the following. The Eq. (8) is completely integrable can be included into an hierarchy of commuting flows. Regular equilibria are those which are common for all these flows [30], while exotic equilibria are fixed points of the particular system (8). Therefore, the technique of the present paper cannot be applied to exotic equilibria (see Remark 3.3). However, if we are able to prove that (8) is non-resonant, then we can assert that all exotic equilibria are unstable (Remark 3.3). Non-resonance was proved in [33], however some technical details in the proof are still missing. A revised proof will be published elsewhere.

7. Several counterexamples

Counterexample 7.1. Let us show that Condition 1 of Lemma 1 cannot be avoided. Let τ be the involution $\lambda \rightarrow -\lambda$, and let $\sigma = -1$. Let $J^2 = \mathrm{diag}(a_1, a_2, a_3)$ where $a_1 < a_2 < a_3$ are real, positive, and distinct. Then the space $\widetilde{\mathcal{P}}_{1,3}(J^2, \tau, \sigma)$ consists of Lax

matrices of the form $M + \lambda J^2$ where $M \in \mathfrak{u}(3)$ is skew-Hermitian. Let

$$M_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then the spectral curve corresponding to $M_0 + \lambda J^2$ is given by

$$P(\lambda, \mu) = (1 + (a_1\lambda - \mu)(a_3\lambda - \mu))(a_2\lambda - \mu) = 0.$$

This curve has a singular point $\lambda_0 = ((a_3 - a_2)(a_2 - a_1))^{-1/2}$, $\mu_0 = a_2\lambda_0$ which does not belong to Fix $\widehat{\tau}$, while all other assumptions of Lemma 1 are satisfied. Let us show that the statement of Lemma 1 does not hold. The gauge group orbit of $M_0 + \lambda J^2$ is

$$\left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ -\bar{y} & 0 & 0 \end{pmatrix} + \lambda J^2 \;\middle|\; |y| = 1 \right\}.$$

The isospectral variety is

$$\left\{ \begin{pmatrix} 0 & z & y \\ -\bar{z} & 0 & x \\ -\bar{y} & -\bar{x} & 0 \end{pmatrix} + \lambda J^2 \;\middle|\; \begin{array}{l} a_1|x|^2 + a_2|y|^2 + a_3|z|^2 = a_2, \\ |x|^2 + |y|^2 + |z|^2 = 1, \\ x\bar{y}z \in \mathbb{R} \end{array} \right\}.$$

It is clear that these two sets are distinct. In particular, the real part of the isospectral variety is the intersection of an ellipsoid with a concentric sphere radius equal to the middle semi-axis of the ellipsoid, i.e. two circles intersecting at two points, while the real part of the gauge group orbit is two points $(0, \pm 1, 0)$.

The statement of Theorem 1 does not hold as well. Let us take

$$\mathcal{L} = \{M + \lambda J^2 \mid M \in \mathfrak{so}(3, \mathbb{R})\}$$

and consider the Lax equation (9), which is the Euler equation of the free three-dimensional rigid body. Then M_0 is the rotation about middle axis of inertia, which is known to be unstable.

Counterexample 7.2. Let us show that the condition on the genus is also important. Consider $\widetilde{\mathcal{P}}_{1,3}(J^2, \tau, \sigma)$ from the previous example, and let

$$M_0 = \begin{pmatrix} 0 & z_0 & y_0 \\ -z_0 & 0 & x_0 \\ -y_0 & -x_0 & 0 \end{pmatrix}$$

where x_0, y_0, z_0 are non-zero real numbers. All conditions of Lemma 1 are satisfied except the condition on the genus: the

spectral curve is non-singular and has genus 1. As in the previous example, take

$$\mathcal{L} = \{M + \lambda J^2 \mid M \in \mathfrak{so}(3, \mathbb{R})\}$$

and consider the Lax equation (9). Then, though the second condition of Theorem 1 is satisfied, the conclusion of the theorem does not hold: M_0 is not a fixed point at all. Consequently, the statement of Lemma 1 does not hold as well. It is actually easy to see that the quotient of the isospectral variety of M_0 by the action of the gauge group is a circle.

Counterexample 7.3. Finally, let us show that Condition 2 of Theorem 2 cannot be avoided as well. Consider $\widetilde{\mathcal{P}}_{1,3}(J^2, \tau, \sigma)$ from previous examples, and let $A = \text{diag}(ai, bi, ci)$ be a diagonal skew-Hermitian matrix. Consider the equation

$$\frac{d}{dt}(M + \lambda J^2) = [M + \lambda J^2, A] \tag{10}$$

which defines a flow on $\mathcal{L} = \widetilde{\mathcal{P}}_{1,3}(J^2, \tau, \sigma)$. Take

$$M_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then Lemma 1 holds true for $M_0 + \lambda J^2$. However, Condition 2 of Theorem 1 is not satisfied, and the statement of the theorem is not true: $M_0 + \lambda J^2$ is not a fixed point of (10). Note that the flow defined by (10) is just the action of a 1-parametric subgroup of the gauge group.

8. Proof of Lemma 1

We need to prove that it is possible to recover L_λ uniquely from its spectral curve up to the action of the gauge group. Let Γ be the normalized compactification of the spectral curve C. Since C is reducible, the Riemann surface Γ is not connected, but has k connected components $\Gamma_1, \ldots, \Gamma_k$. The functions λ, μ are meromorphic on Γ and satisfy the equation $\det(L_\lambda - \mu E) = 0$. Let also $\pi \colon \Gamma \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the mapping $\gamma \mapsto (\lambda(\gamma), \mu(\gamma))$. The image of π is the closure of the curve C.

Proposition 8.1. *The meromorphic function λ has exactly n poles $\infty_1, \ldots, \infty_n$ in Γ.*

Proof. This follows easily from the simplicity of the spectrum of B_m. □

Let $\Gamma_0 = \{\gamma \in \Gamma \mid \lambda(\gamma) < \infty, \dim \text{Ker}\,(L_{\lambda(\gamma)} - \mu(\gamma)E) = 1\}$. Consider the mapping $\psi \colon \Gamma_0 \to \mathbb{CP}^{n-1}$ which maps γ to the eigenvector of $L_{\lambda(\gamma)} - \mu(\gamma)E$. This mapping defines a holomorphic line bundle over Γ_0 which is called the *eigenvector bundle*. However, we will consider ψ as a mapping, not as a line bundle. The following statement is well-known (see e.g. Audin [5]).

Proposition 8.2. *The mapping ψ can be uniquely extended to a holomorphic mapping $\psi \colon \Gamma \to \mathbb{CP}^{n-1}$.*

Proof. The complement $\Gamma \setminus \Gamma_0$ consists of the finite number of points $\gamma_1, \ldots, \gamma_k$. Prove that ψ can be extended to γ_i. Let z be a local coordinate in the neighborhood of γ_i such that $z(\gamma_i) = 0$. Without loss of generality, the first row of the comatrix of $L_{\lambda(z)} - \mu(z)E$ is non-zero and finite in the punctured neighborhood of γ_i. Denote this vector by $a(z) = (a^1(z), \ldots, a^n(z))$. Clearly,

$$\psi(z) = (a^1(z) : \ldots : a^n(z))$$

for $z \neq 0$. At the same time, there exists a positive or negative integer m such that $a(z) = z^m b(z)$ where $b(z) = (b_1(z), \ldots,$

$b_n(z)) \neq 0$ at $z = 0$. So, ψ can be extended to γ_i by the formula

$$\psi(z) = (b^1(z) : \ldots : b^n(z)).$$

Uniqueness is obvious. □

Proposition 8.3. *The vectors $\psi(\infty_1), \ldots, \psi(\infty_n)$ are the eigenvectors of B_m.*

Proof. Apply the continuity argument. □

Let us normalize ψ to obtain a meromorphic vector-function $h \colon \Gamma \to \mathbb{C}^n$ such that $h(\gamma) \neq 0$ for each $\gamma \in \Gamma$:

$$h^i = \psi^i \left(\sum_{i=1}^n \alpha_i \psi^i \right)^{-1}.$$

The numbers $\alpha_1, \ldots, \alpha_n$ should be chosen in such a way that the poles of h are away from the ramification points of λ, as well from the points which project to singular points of the curve C.

Count the number of poles of h. For generic $\lambda_0 \in \mathbb{CP}^1$, the set $\lambda^{-1}(\lambda_0) = \{\gamma \in \Gamma \mid \lambda(\gamma) = \lambda\}$ contains exactly n points $\gamma_1, \ldots, \gamma_n$. Following Dubrovin et al. [1], consider the function

$$r(\lambda_0) = (\det(h(\gamma_1), \ldots, h(\gamma_n)))^2.$$

This expression does not depend on the numeration of points in $\lambda^{-1}(\lambda_0)$, so it is a rational function of λ_0. Clearly,

$$(r)_\infty = 2\lambda\,((h)_\infty) \tag{11}$$

where $(f)_\infty$ denotes the divisor of poles of f. Therefore, we can count poles of h by counting poles of r. To count poles of r, we count its zeros. Obviously, $r(\lambda_0)$ can only be zero if the spectrum of L_{λ_0} is not simple. This may happen in two cases: either if λ_0 is a branch point of the function λ, or if the preimage $\lambda^{-1}(\lambda_0)$ contains singular points of C, i.e. those points which belong to at least two irreducible components. Below we count the number of zeros of r corresponding to branch points and show that Condition 1 of Lemma 1 guarantees that singular points *do not* contribute to zeros of r.

Proposition 8.4. *Let $\gamma \in \Gamma \setminus \{\infty_1, \ldots, \infty_n\}$, and let z be a local parameter near γ. Then either $\lambda_z' \neq 0$, or $\mu_z' \neq 0$.*

Proof. The set $\Gamma \setminus \{\infty_1, \ldots, \infty_n\}$ is just the disjoint union of irreducible components of C, so the statement follows from the smoothness of irreducible components. □

Proposition 8.5. *Let $\gamma \in \Gamma \setminus \{\infty_1, \ldots, \infty_n\}$, and let z be a local parameter near γ. Assume that $\lambda_z'(\gamma) = 0$. Then the matrix $L_{\lambda(\gamma)}$ has a non-trivial Jordan block with eigenvalue $\mu(\gamma)$ and generalized eigenvector $h_z'(\gamma)$.*

Proof. Without loss of generality, assume that $z(\gamma) = 0$. Differentiating

$$(L_{\lambda(z)} - \mu(z)E)h(z) = 0$$

with respect to z at $z = 0$, obtain

$$(L_{\lambda(0)} - \mu(0)E)h'(0) = \mu'(0)h(0).$$

By Proposition 8.4, the number $\mu'(0)$ is non-zero, which proves the proposition. □

Proposition 8.6. *Assume that $\gamma_1, \ldots, \gamma_p \in \Gamma$ are distinct, and that $\pi(\gamma_i) = \pi(\gamma_j)$ for any $i, j \in \{1, \ldots, p\}$. Then the vectors $h(\gamma_1), \ldots, h(\gamma_p)$ are linearly independent.*

Proof. Let $\lambda(\gamma_i) = \lambda_0$, and $\mu(\gamma_i) = \mu_0$. Then $(\lambda_0, \mu_0) \in C$ is a singular point. Condition 1 of Lemma 1 implies that L_{λ_0} is a normal operator (recall that an operator in Hermitian space is called normal if it commutes with its adjoint), therefore it has no non-trivial Jordan blocks. Using Propositions 8.4 and 8.5, we conclude that λ can be taken as a local parameter near $\gamma_1, \ldots, \gamma_p$.

Let $\mu = \mu_i(\lambda)$ and $h = h_i(\lambda)$ in the neighborhood of γ_i. Assume that

$$\sum_{i=1}^{p} c_i h_i(\lambda_0) = 0 \tag{12}$$

and prove that $c_i = 0$ for each i. Differentiating the equation

$$(L_\lambda - \mu_i(\lambda) E) h_i(\lambda) = 0$$

with respect to λ at $\lambda = \lambda_0$, we have

$$Q_i = \left(L'_\lambda(\lambda_0) - \mu'_i(\lambda_0) \right) h_i(\lambda_0) + (L_{\lambda_0} - \mu_0 E) h'_i(\lambda_0) = 0. \tag{13}$$

Using (12), we obtain

$$0 = \sum_{i=1}^{p} c_i Q_i = -\sum_{i=1}^{p} c_i \mu'_i(\lambda_0) h_i(\lambda_0)$$
$$+ (L_{\lambda_0} - \mu_0 E) \left(\sum_{i=1}^{p} c_i h'_i(\lambda_0) \right),$$

so

$$(L_{\lambda_0} - \mu_0 E) \left(\sum_{i=1}^{p} c_i h_i(\lambda_0) \right) = \sum_{i=1}^{p} c_i \mu'_i(\lambda_0) h_i(\lambda_0). \tag{14}$$

Since L_{λ_0} has no non-trivial Jordan blocks, (14) implies that

$$\sum_{i=1}^{p} c_i \mu'_i(\lambda_0) h_i(\lambda_0) = 0.$$

Continuing in the same fashion, we obtain

$$\sum_{i=1}^{p} c_i (\mu'_i(\lambda_0))^k h_i(\lambda_0) = 0 \quad \forall k \in \mathbb{N} \cup \{0\}. \tag{15}$$

Let $I = \{1, \ldots, p\}$. Define an equivalence relation on I by $i \equiv j \Leftrightarrow \mu'_i(\lambda_0) = \mu'_j(\lambda_0)$. Denote equivalence classes by $I_1, \ldots, I_q$. Then (15) implies that

$$\sum_{i \in I_s} c_i h_i(\lambda_0) = 0, \quad \forall s = 1, \ldots, q. \tag{16}$$

If all pairwise intersections of irreducible components of C are of order 1, then each class I_s consists of one element, and we are done. Otherwise, differentiate

$$(L_\lambda - \mu_i(\lambda) E) h_i(\lambda) = 0$$

two times with respect to λ at $\lambda = \lambda_0$:

$$S_i = (L''_{\lambda_0} - \mu''_i(\lambda_0) E) h_i(\lambda_0) + 2(L'_{\lambda_0} - \mu'_i(\lambda_0) E) h'_i(\lambda_0)$$
$$+ (L_{\lambda_0} - \mu_0 E) h''_i(\lambda_0) = 0.$$

Fixing $s \in \{1, \ldots, q\}$ and using (16), we get

$$0 = \sum_{i \in I_s} c_i S_i = -\sum_{i \in I_s} c_i \mu''_i(\lambda_0) h_i(\lambda_0)$$
$$+ 2(L'_{\lambda_0} - \mu'(\lambda_0) E) \sum_{i \in I_s} c_i h'_i(\lambda_0)$$
$$+ (L_{\lambda_0} - \mu_0 E) \sum_{i \in I_s} c_i h''_i(\lambda_0) \tag{17}$$

where $\mu'(\lambda_0) = \mu'_i(\lambda_0)$ for $i \in I_s$.

Let $L = L_{\lambda_0} - \mu_0 E$, and let $L' = L'_{\lambda_0} - \mu'(\lambda_0) E$. Then it is easy to see that

$$L^* = \alpha L, \quad \text{and} \quad (L')^* = \beta L' \tag{18}$$

where $\alpha = \sigma^{-1}$, and $\beta \in \mathbb{C}$ is some constant. Let $\langle \, , \rangle$ be the standard Hermitian scalar product $\langle x, y \rangle = x^* y$. Using (17), we have

$$\left\langle \sum_{i \in I_s} c_i \mu''_i(\lambda_0) h_i(\lambda_0), h_j(\lambda_0) \right\rangle = X_{sj} + Y_{sj} \tag{19}$$

where

$$X_{sj} = 2 \left\langle L' \left(\sum_{i \in I_s} c_i h'_i(\lambda_0) \right), h_j(\lambda_0) \right\rangle,$$

$$Y_{sj} = \left\langle L \left(\sum_{i \in I_s} c_i h''_i(\lambda_0) \right), h_j(\lambda_0) \right\rangle.$$

Using (18), we show that Y_{sj} vanishes:

$$Y_{sj} = \alpha \left\langle \sum_{i \in I_s} c_i h''_i(\lambda_0), L h_j(\lambda_0) \right\rangle = 0.$$

Using (18) together with (13) and (16), we show that X_{sj} is also zero:

$$X_{sj} = 2\beta \left\langle \sum_{i \in I_s} c_i h'_i(\lambda_0), L' h_j(\lambda_0) \right\rangle = -2\beta \left\langle \sum_{i \in I_s} c_i h'_i(\lambda_0), L h'_j(\lambda_0) \right\rangle$$

$$= -2\beta\bar{\alpha} \left\langle L \left(\sum_{i \in I_s} c_i h'_i(\lambda_0) \right), h'_j(\lambda_0) \right\rangle$$

$$= 2\beta\bar{\alpha} \left\langle L' \left(\sum_{i \in I_s} c_i h_i(\lambda_0) \right), h'_j(\lambda_0) \right\rangle = 0.$$

Using (19) we conclude that

$$\left\langle \sum_{i \in I_s} c_i \mu''_i(\lambda_0) h_i(\lambda_0), h_j(\lambda_0) \right\rangle = 0$$

for any s and j, hence

$$\sum_{i \in I_s} c_i \mu''_i(\lambda_0) h_i(\lambda_0) = 0 \quad \forall s = 1, \ldots, q.$$

Continuing in the same fashion, we obtain

$$\sum_{i \in I_s} c_i (\mu''_i(\lambda_0))^k h_i(\lambda_0) = 0 \quad \forall s = 1, \ldots, q, \; k \in \mathbb{N} \cup \{0\}. \tag{20}$$

Since all pairwise intersections of irreducible components of C are of order at most two, we have $\mu''_i(\lambda_0) \neq \mu''_j(\lambda_0)$ for any distinct $i, j \in I_s$. Consequently, (20) implies that $c_i = 0$ for each i. $\quad \square$

Proposition 8.7. *The divisor of zeros of r is given by $(r)_0 = \lambda((\lambda)_R)$ where $(\lambda)_R$ is the ramification divisor of λ.*

Proof. This statement is well-known in the case when the spectral curve is non-singular [1], and as follows from Proposition 8.6, singular points do not contribute to $(r)_0$. If the set $\lambda^{-1}(\lambda_0)$ contains n distinct points, then Proposition 8.6 implies that $r(\lambda_0) \neq 0$. So, $r(\lambda_0) = 0$ if and only if λ_0 is a branch point. For simplicity, assume that the set $\lambda^{-1}(\lambda_0)$ contains exactly one simple ramification point of λ. Let z be a local coordinate on Γ such that $\lambda - \lambda_0 = z^2$.

Then

$$r(\lambda) = \big(\det(\dots, h(0) + h'(0)z + O(z^2), h(0)$$
$$- h'(0)z + O(z^2), \dots)\big)^2$$
$$= \big(\det(\dots, 2h'(0)z + O(z^2), h(0) + O(z), \dots)\big)^2$$
$$= (\lambda - \lambda_0)\big(\det(\dots, 2h'(0) + O(z), h(0) + O(z), \dots)\big)^2.$$

By Proposition 8.5, the vector $h'(0)$ is a generalized eigenvector for L_{λ_0}, so the latter determinant is non-zero, which proves that $r(\lambda)$ has a simple zero at $\lambda = \lambda_0$. The proof in the general case is analogous. $\square$

Since r is a meromorphic function, $\deg(r)_0 = \deg(r)_\infty$. Using (11) and Proposition 8.7, we have

$$\deg(h)_\infty = \frac{1}{2}\deg(\lambda)_R,$$

or, by Riemann–Hurwitz formula,

$$\deg(h)_\infty = n - k + \sum_{i=1}^{k} g_i,$$

where g_i is the genus of Γ_i. If $g_i = 0$ for each i, then

$$\deg(h)_\infty = n - k. \tag{21}$$

Let $\xi_1, \dots, \xi_n$ be the eigenvectors of B ordered in such a way that $h(\infty_i) = \xi_i$, and

$$\infty_1, \dots, \infty_{n_1} \in \Gamma_1, \qquad \infty_{n_1+1}, \dots, \infty_{n_1+n_2} \in \Gamma_2, \quad \dots.$$

Denote by $h^1, \dots, h^n$ the components of h in the basis $\xi_1, \dots, \xi_n$: $h = \sum h^i \xi_i$. Then $h^1(\infty_2) = \dots = h^1(\infty_{n_1}) = 0$, but $h^1(\infty_1) \neq 0$. Consequently,

$$\deg(h|_{\Gamma_1})_\infty \geq \deg(h^1|_{\Gamma_1})_\infty = \deg(h^1|_{\Gamma_1})_0 \geq n_1 - 1.$$

Analogously,

$$\deg(h|_{\Gamma_i})_\infty \geq n_i - 1 \quad \text{for } i = 1, \dots, k.$$

Comparing with (21), we conclude that

$$\deg(h|_{\Gamma_i})_\infty = n_i - 1. \tag{22}$$

Identify each Γ_i with the standard Riemann sphere, and renormalize h in such a way that all coordinates of $h|_{\Gamma_i}$ are polynomials of degree $n_i - 1$. Each of these coordinates vanishes at exactly $n_i - 1$ points which are fixed, so all coordinates of h are defined uniquely modulo a constant factor. This means that L_λ can be recovered uniquely up to conjugation by a matrix which commutes with the leading term B. Let us show that this matrix can be chosen to be unitary.

Let $L_\lambda = Q^{-1}L_\lambda^0 Q$. Then $L_\lambda^* = Q^*(L_\lambda^0)^*(Q^{-1})^*$ which can be rewritten as

$$L_{\tau(\lambda)} = Q^* L_{\tau(\lambda)}^0 (Q^{-1})^*. \tag{23}$$

On the other hand,

$$L_{\tau(\lambda)} = Q^{-1} L_{\tau(\lambda)}^0 Q. \tag{24}$$

Comparing (23) and (24), we conclude that L_λ^0 commutes with QQ^* for any λ. Let $S = \sqrt{QQ^*}$. Then the polar decomposition of Q is $Q = SU$ where U is unitary. Since QQ^* commutes with L_λ^0, so does S, therefore

$$L_\lambda = U^{-1} L_\lambda^0 U.$$

Obviously, U commutes with the leading term B, so it belongs to the gauge group G. $\square$

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