Change of Polytope Volumes Under Möbius Transformations and the Circumcenter Of Mass

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Abstract

The circumcenter of mass of a simplicial polytope P is defined as follows: triangulate P, assign to each simplex its circumcenter taken with weight equal to the volume of the simplex, and then find the center of mass of the resulting system of point masses. The so obtained point is independent of the triangulation. The aim of the present note is to give a definition of the circumcenter of mass that does not rely on a triangulation. To do so we investigate how volumes of polytopes change under Möbius transformations.

1 Introduction

Recall that the center of mass of a polyhedral solid P can be found as follows: triangulate P, assign to each simplex its centroid taken with weight equal to the volume of the simplex, and then find the center of mass of the resulting system of point masses. The so obtained point is independent of the triangulation.

Remarkably, when P is simplicial (i.e., all its facets are simplices), one can replace centroids of simplices in this construction by their circumcenters. The resulting point still does not depend on the triangulation and is known as the *circumcenter of mass* of P. The author of [4] attributes this construction to 19th century algebraic geometer

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Giusto Bellavitis. In modern literature, the circumcenter of mass is studied in [1, 3, 6–8], as well is in [5] for a special class of polyhedra whose edges touch a sphere.

Since the circumcenter of mass does not depend on the triangulation, one should be able to define it without using one. The aim of the present note is to provide such a definition. Specifically, we show that the circumcenter of mass of P is related to the rate of change of volume of P under Möbius transformations.

Recall that a Möbius transformation of $\mathbb{R}^n$ is an isometry of the hyperbolic upper half-space $\mathbb{H}^{n+1}$ restricted to the boundary. Every such transformation is a finite composition of inversions in spheres and reflections in hyperplanes. For $n \geq 3$ Möbius transformations are the same as conformal transformations, while for $n = 2$ Möbius transformations are just complex fractional linear transformations $z \mapsto (az + b)/(cz + d)$.

Note that general Möbius transformations do not preserve coplanarity and hence polyhedral shapes. However, the action of Möbius transformations on *simplicial* polytopes is still well-defined. By definition, a Möbius transformation $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ takes a simplex Δ with vertices $v_0, \ldots, v_n$ to a simplex $\phi(\Delta)$ with vertices $\phi(v_0), \ldots, \phi(v_n)$. This extends to simplicial polytopes: the image under ϕ of a polytope with faces $\Delta_1, \ldots, \Delta_k$ is the polytope with faces $\phi(\Delta_1), \ldots, \phi(\Delta_k)$.

Consider a simplicial polytope P. How does its volume change under infinitesimal Möbius transformations? We show that this is determined by the location of one single point of P, that we call the *Möbius center* of P and denote as m(P) (see Theorem 2.1 (i)). Specifically, *the relative rate of change of volume under an infinitesimal Möbius transformation ξ is equal to the divergence of ξ computed at the Möbius center*:

$$\nabla_\xi \log \mathrm{vol}(P) = \mathrm{div}\, \xi(\mathrm{m}(P)). \tag{1}$$

Here ∇_ξ stands for the derivative in the direction ξ:

$$\nabla_\xi \log \mathrm{vol}(P) = \left.\frac{d}{dt}\right|_{t=0} \log \mathrm{vol}(\phi_t(P)),$$

where ϕ_t is a family of Möbius transformations integrating ξ.

Note that Möbius vector fields are quadratic (see formula (4)) and hence have linear divergence. So, formula (1) implies that, just like the center and circumcenter of mass, the Möbius center m(P) can be found by subdivision into simplices (see Theorem 2.1 (iii)). Furthermore, we show that for a simplex Δ its Möbius center m(Δ) coincides with the circumcenter of its medial simplex Δ', i.e., the simplex whose vertices are centroids of facets (codimension 1 faces) of Δ (see Theorem 2.1 (iv)). So, the Möbius center can be defined by the same construction as the circumcenter of mass, with circumcenters of simplices of the triangulation replaced by circumcenters of their medial simplices. Furthermore, there is a simple relation between the two circumcenters: for a n-dimensional simplex Δ one has

$$\mathrm{cc}(\Delta') = \frac{n+1}{n}\,\mathrm{cm}(\Delta) - \frac{1}{n}\,\mathrm{cc}(\Delta), \tag{2}$$

where Δ' is the medial simplex, cm stands for the center of mass, and cc for the circumcenter. As a result, for a general simplicial n-dimensional polytope P one has

$$\mathrm{m}(P) = \frac{n+1}{n}\,\mathrm{cm}(P) - \frac{1}{n}\,\mathrm{ccm}(P),$$

where $\mathrm{ccm}(P)$ is the circumcenter of mass of P. So, since both the center of mass and the Möbius center can be defined without a triangulation, it follows that the circumcenter of mass is well defined as well. Explicitly, one has

$$\mathrm{ccm}(P) = (n+1)\,\mathrm{cm}(P) - n \cdot \mathrm{m}(P). \tag{3}$$

Remark 1.1 One may similarly ask how the volume of a polytope changes under infinitesimal *projective* transformations. In that case, one has the following version of formula (1): $\nabla_\xi \log \mathrm{vol}(P) = \mathrm{div}\,\xi(\mathrm{cm}(P))$. Indeed, since projective transformations take faces to faces, the change of volume of P under an infinitesimal projective transformation ξ can be computed as the flux of ξ through the boundary of P or, equivalently, as the integral of the divergence of ξ over the interior of P (for simplicity assume that P is convex):

$$\nabla_\xi \log \mathrm{vol}(P) = \frac{1}{\mathrm{vol}(P)} \int_{\text{interior of } P} \mathrm{div}\,\xi\,\mathrm{d}x,$$

where $\mathrm{d}x$ is the Euclidean volume element. But since the divergence of a projective vector field is a (inhomogeneous) linear function, the latter expression is precisely the value of the divergence at the center of mass, as needed.

Remark 1.2 Formula (3) may seem unsettling as the coefficients look pretty random. There is a way to fix this by replacing Möbius vector fields by a different class of quadratic fields ξ which in some sense interpolate between Möbius and projective fields (see Remark 2.1). Those vector fields have the property $\nabla_\xi \log \mathrm{vol}(P) = \mathrm{div}\,\xi(\mathrm{ccm}(P))$ and thus provide a direct definition of the circumcenter of mass circumventing the notion of the Möbius center. However, the geometric meaning of such fields ξ is somewhat unclear.

Remark 1.3 For a triangle, the circumcenter of the medial triangle (i.e., the Möbius center) is also known as the *nine-point center*. So, analogously to the definition of the circumcenter of mass, the Möbius center of a polygon can be thought of as the "nine-point center of mass". An analogous point of the tetrahedron is the center of the so-called *twelve-point sphere*, however it does not seem to have any name. Relation (2) is well known in those cases. In particular, for a triangle it says that the centroid lies on the line joining the circumcenter and the nine-point center, 2/3 of the way towards the latter (the line containing all the three points is known as the *Euler line*; it also contains the orthocenter).

2 Precise Definitions and the Main Result

There are many ways to formalize the notion of a (not necessarily convex) polytope. For the purposes of the present paper, a *simplicial polytope* in $\mathbb{R}^n$ is a piecewise linear simplicial cycle of dimension $n - 1$ (a particular case of this general definition is the boundary of a convex polytope all of whose facets are simplices). In particular, polytopes in $\mathbb{R}^n$ form an Abelian group $\mathcal{P}(\mathbb{R}^n)$ under addition. This group is generated by boundaries of oriented n-dimensional simplices (in what follows, we refer to those generators as just simplices). A representation of a polytope P as a sum of simplices is called a *triangulation*.

The group of polytopes comes equipped with the (algebraic) volume homomorphism vol: $\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$. It is given on simplices $\Delta = (v_0, \ldots, v_n)$ by vol$(\Delta) = |v_1 - v_0, \ldots, v_n - v_0|/n!$, where $|w_1, \ldots, w_n|$ stands for the determinant of the matrix $(w_1, \ldots, w_n)$.

We define *Möbius transformations* of $\mathbb{R}^n$ as isometries of the hyperbolic upper half-space $\mathbb{H}^{n+1}$ restricted to the boundary. We refer the reader to [2] for a detailed account of such transformations. Here we only need the corresponding Lie algebra $\mathfrak{möb}_n$ of Möbius vector fields. In dimensions $n \geq 3$, Möbius vector fields are the same as conformal Killing vector fields. In dimension $n = 2$, they are the same as holomorphic quadratic vector fields. In any dimension, the general form of a Möbius vector field ξ is

$$\dot{x} = Ax + |x|^2 b - 2\langle b, x \rangle x + c, \tag{4}$$

where A is a matrix such that $A - \lambda \operatorname{Id}$ is skew-symmetric for some $\lambda \in \mathbb{R}$, and $b, c \in \mathbb{R}^n$ are vectors. We note that the divergence of such a vector field is

$$\operatorname{div} \xi = \operatorname{tr} A - 2n \langle b, x \rangle. \tag{5}$$

Theorem 2.1 (i) *Let $P \in \mathcal{P}(\mathbb{R}^n)$ be a polytope in $\mathbb{R}^n$ with vol$(P) \neq 0$. Then there exists a unique point $\mathrm{m}(P) \in \mathbb{R}^n$ (the Möbius center of $\mathbf{P}$) such that*

$$\nabla_\xi \log \operatorname{vol}(P) = \operatorname{div} \xi(\mathrm{m}(P)). \tag{6}$$

for any Möbius vector field $\xi \in \mathfrak{möb}_n$.

(ii) *For P as above and any similarity transformation $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ (i.e., a composition of a homothety and isometry) one has $\mathrm{m}(\phi(P))) = \phi(\mathrm{m}(P))$.*

(iii) *For P as above and any triangulation $P = \sum \Delta_i$ with vol$(\Delta_i) \neq 0$ for all i one has*

$$\mathrm{m}(P) = \frac{1}{\operatorname{vol}(P)} \sum \operatorname{vol}(\Delta_i) \, \mathrm{m}(\Delta_i).$$

(iv) *For a simplex $\Delta \in \mathcal{P}(\mathbb{R}^n)$ such that vol$(\Delta) \neq 0$, the Möbius center $\mathrm{m}(\Delta)$ coincides with the circumcenter of the medial simplex Δ'. It is related to the centroid and*

the circumcenter of Δ by the formula

$$\mathrm{cc}(\Delta') = \frac{n+1}{n}\,\mathrm{cm}(\Delta) - \frac{1}{n}\,\mathrm{cc}(\Delta). \tag{7}$$

(v) *For any $P \in \mathcal{P}(\mathbb{R}^n)$ with $\mathrm{vol}(P) \neq 0$ its Möbius center is related to the center of mass and the circumcenter of mass by the formula*

$$\mathrm{m}(P) = \frac{n+1}{n}\,\mathrm{cm}(P) - \frac{1}{n}\,\mathrm{ccm}(P),$$

Proof (i) Consider the subalgebra $\mathfrak{iso}_n \subset \mathfrak{m\ddot{o}b}_n$ of infinitesimal isometries. It consists of vector fields of the form (4) with $b = 0$ and A skew-symmetric. We have the following sequence of linear maps:

$$0 \to \mathfrak{iso}_n \xrightarrow{i} \mathfrak{m\ddot{o}b}_n \xrightarrow{\mathrm{div}} \mathfrak{l}_n \to 0$$

where $i : \mathfrak{iso}_n \to \mathfrak{m\ddot{o}b}_n$ is the inclusion mapping, and $\mathfrak{l}_n$ is the $(n+1)$-dimensional vector space of (inhomogeneous) linear functions on the affine space $\mathbb{R}^n$. Since every divergence-free Möbius vector field is an isometry (preserving angles and volume implies preserving the metric), and every linear function can be obtained as the divergence of a Möbius vector field (which follows from transitivity of action of isometries on linear functions and can also be seen from explicit expression (5)), this sequence is exact. Therefore, any linear function on $\mathfrak{m\ddot{o}b}_n$ which vanishes on $\mathfrak{iso}_n$ is of the form $f(\mathrm{div}\,\xi)$, where $f \in \mathfrak{l}_n^*$ (here $\mathfrak{l}_n^*$ is the dual space of $\mathfrak{l}_n$). This in particular applies to the function $\xi \mapsto \nabla_\xi \log \mathrm{vol}(P)$ (which vanishes on isometries since isometries preserve the volume). So, there is $f \in \mathfrak{l}_n^*$ such that

$$\nabla_\xi \log \mathrm{vol}(P) = f(\mathrm{div}\,\xi).$$

for all $\xi \in \mathfrak{m\ddot{o}b}_n$. Further observe that for ξ of the form $\dot{x} = \lambda x$ (i.e., a homothety) one has $\nabla_\xi \log \mathrm{vol}(P) = n\lambda$ and $\mathrm{div}\,\xi = n\lambda$. Therefore, for any constant function $c \in \mathfrak{l}_n$ one has $f(c) = c$. But any linear function $f : \mathfrak{l}_n \to \mathbb{R}$ which takes every constant to itself is of the form $f(l) = l(x)$ for some $x \in \mathbb{R}^n$. Furthermore, such x is clearly unique, since evaluation at different points gives different functions on $\mathfrak{l}_n$ Denoting that x by $\mathrm{m}(P)$, we get the result.

(ii) This follows from the invariance of all involved objects under similarities.

(iii) This follows from additivity of the volume function and linearity of $\mathrm{div}\,\xi$ for a Möbius vector field $\xi \in \mathfrak{m\ddot{o}b}_n$:

$$\nabla_\xi \log \mathrm{vol}(\Delta_i) = \mathrm{div}\,\xi(\mathrm{m}(\Delta_i)) \implies \nabla_\xi \mathrm{vol}(\Delta_i) = \mathrm{vol}(\Delta_i)\,\mathrm{div}\,\xi(\mathrm{m}(\Delta_i))$$

$$\implies \nabla_\xi \mathrm{vol}(P) = \sum \nabla_\xi \mathrm{vol}(\Delta_i) = \sum \mathrm{vol}(\Delta_i)\,\mathrm{div}\,\xi(\mathrm{m}(\Delta_i))$$

$$\implies \nabla_\xi \ln \mathrm{vol}(P) = \sum \frac{\mathrm{vol}(\Delta_i)}{\mathrm{vol}(P)}\,\mathrm{div}\,\xi(\mathrm{m}(\Delta_i)) = \mathrm{div}\,\xi\left(\sum \frac{\mathrm{vol}(\Delta_i)}{\mathrm{vol}(P)}\,\mathrm{m}(\Delta_i)\right).$$

(iv) Let $\Delta = (v_0, \ldots, v_n) \in \mathcal{P}(\mathbb{R}^n)$ be a simplex such that $\mathrm{vol}(\Delta) \neq 0$, and let

$$p := -n \cdot \mathrm{m}(\Delta) + \sum_{i=0}^{n} v_i. \tag{8}$$

We will first show that p is the circumcenter of Δ. In view of part (i) of the theorem, it suffices to consider the case $v_0 = 0$. Let $\xi \in \mathfrak{m\ddot{o}b}_n$ be of the form

$$\dot{x} = |x|^2 b - 2 \langle b, x \rangle x,$$

and let ϕ_t be a family of Möbius transformations integrating ξ. Note that since ξ vanishes at the origin, we have $\phi_t(v_0) = \phi_t(0) = 0$. Therefore,

$$
\begin{aligned}
\nabla_\xi \log \mathrm{vol}(\Delta) &= \frac{d}{dt}\bigg|_{t=0} \log |\phi_t(v_1), \ldots, \phi_t(v_n)| \\
&= \frac{1}{D} \sum_{i=1}^{n} |v_1, \ldots, |v_i|^2 b - 2\langle b, v_i \rangle i, \ldots, v_n| \\
&= \ell(b) - 2 \left\langle b, \sum_{i=1}^{n} v_i \right\rangle,
\end{aligned}
$$

where

$$D := |v_1, \ldots, v_n|, \quad \ell(b) := \frac{1}{D} \sum_{i=1}^{n} |v_1, \ldots, b, \ldots, v_n| \cdot |v_i|^2. \tag{9}$$

On the other hand, we have $\nabla_\xi \log \mathrm{vol}(\Delta) = \mathrm{div}\, \xi(\mathrm{m}(\Delta))$ and $\mathrm{div}\, \xi = -2n \langle b, x \rangle$, so by (6) we have

$$-2n \langle b, \mathrm{m}(\Delta) \rangle = \ell(b) - 2 \left\langle b, \sum_{i=1}^{n} v_i \right\rangle$$

for any $b \in \mathbb{R}^n$, which, by (8), is equivalent to $\langle b, p \rangle = \ell(b)/2$. In particular, by definition (9) of the function ℓ we get

$$\langle p, v_i \rangle = \frac{\ell(v_i)}{2} = \frac{\langle v_i, v_i \rangle}{2}.$$

Therefore,

$$|p - v_i|^2 = \langle p, p \rangle - 2\langle p, v_i \rangle + \langle v_i, v_i \rangle = \langle p, p \rangle = |p - v_0|^2 \tag{10}$$

for all i. So indeed p is the circumcenter of Δ, as claimed.

Now, let us show that $m(\Delta)$ is the circumcenter of the medial simplex of Δ. In view of (8) and an already established fact that p is the circumcenter of Δ, this also proves (7). Let

$$v_i' := \frac{1}{n}\left(-v_i + \sum_{j=0}^{n} v_j\right)$$

be centroids of facets of Δ (equivalently, vertices of the medial simplex Δ'). Then

$$|v_i' - m(\Delta)|^2 = \frac{1}{n^2}\left|-v_i + \sum_{j=0}^{n} v_j - n \cdot m(\Delta)\right|^2 = \frac{|p - v_i|^2}{n^2}.$$

By (10), the latter quantity is independent of i, so $m(\Delta)$ is equidistant from the points $v_0', \ldots, v_n'$, as desired.

(v) This directly follows from the two previous statements. □

Remark 2.1 Along the same lines one shows that for quadratic vector fields ξ of the form

$$\dot{x} = Ax + |x|^2 b + c \tag{11}$$

one has $\nabla_\xi \log \text{vol}(P) = \text{div} \, \xi(\text{ccm}(P))$. This gives a direct definition of the circumcenter of mass bypassing the notion of the Möbius center. We chose not to pursue this approach since geometric interpretation of vector fields (11) is unclear. In a certain sense such vector fields interpolate between Möbius fields (4) and projective fields $\dot{x} = Ax + \langle b, x\rangle x + c$, just like the circumcenter of mass "interpolates" between the center of mass and the Möbius center.

References

1. Adler, V.E.: Recuttings of polygons. Funct. Anal. Appl. **27**(2), 141–143 (1993)
2. Ahlfors, L.V.: Möbius Transformations in Several Dimensions. Ordway Professorship Lectures in Mathematics. School of Mathematics, University of Minnesota, Minneapolis (1981)
3. Akopyan, A.V.: Some remarks on the circumcenter of mass. Discrete Comput. Geom. **51**(4), 837–841 (2014)
4. Laisant, C.-A.: Theorie et Applications des Équipollences. Gauthier-Villars, Paris (1887)
5. Lángi, Zs.: Centering Koebe polyhedra via Möbius transformations. Groups Geom. Dyn. **15**(1), 197–221 (2021)
6. Myakishev, A.: On two remarkable lines related to a quadrilateral. Forum Geom. **6**, 289–295 (2006)

7. Tabachnikov, S., Tsukerman, E.: Circumcenter of mass and generalized Euler line. Discrete Comput. Geom. **51**(4), 815–836 (2014)
8. Tabachnikov, S., Tsukerman, E.: Remarks on the circumcenter of mass. Arnold Math. J. **1**(2), 101–112 (2015)