

The Limit Point of the Pentagon Map and Infinitesimal Monodromy

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The pentagram map takes a planar polygon P to a polygon P' whose vertices are the intersection points of the consecutive shortest diagonals of P . The orbit of a convex polygon under this map is a sequence of polygons that converges exponentially to a point. Furthermore, as recently proved by Glick, coordinates of that limit point can be computed as an eigenvector of a certain operator associated with the polygon. In the present paper, we show that Glick's operator can be interpreted as the *infinitesimal monodromy* of the polygon. Namely, there exists a certain natural infinitesimal perturbation of a polygon, which is again a polygon but in general not closed; what Glick's operator measures is the extent to which this perturbed polygon does not close up.

1 Introduction

The pentagram map, introduced by Schwartz [10], is a discrete dynamical system on the space of planar polygons. The definition of this map is illustrated in Figure 1: the image of the polygon P under the pentagram map is the polygon P' whose vertices are the intersection points of the consecutive shortest diagonals of P (i.e., diagonals connecting 2nd-nearest vertices).

The pentagram map has been an especially popular topic in the past decade, mainly due to its connections with integrability [8, 12] and the theory of cluster algebras [2–4]. Most works on the pentagram map regard it as a dynamical system on the space

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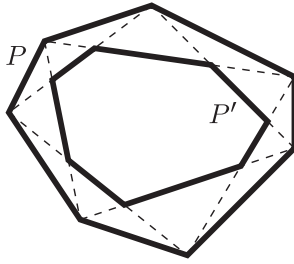


Fig. 1. The pentagram map.

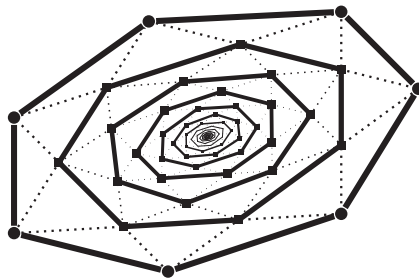


Fig. 2. The orbit of a convex polygon under the pentagram map converges to a point.

of polygons modulo projective equivalence. And indeed, that is the setting where most remarkable features of that map such as integrability reveal themselves. That said, the pentagram map on actual polygons (as opposed to projective equivalence classes) also has interesting geometry. One of the early results in this direction was Schwartz's proof of the exponential convergence of successive images of a convex polygon under the pentagram map to a point (Figure 2). That limit point is a natural invariant of a polygon and can be thought of as a projectively natural version of the center of mass. However, it is not clear a priori whether this limit point can be expressed in terms of coordinates of the vertices by any kind of an explicit formula. A remarkable recent result by Glick [5] is that this dependence is in fact algebraic. Moreover, there exists an operator in \mathbb{R}^3 whose matrix entries are rational in terms of polygon's vertices, while the coordinates of the limit point are given by an eigenvector of that operator. Therefore, the coordinates of the limit point can be found by solving a cubic equation.

Specifically, suppose we are given an n -gon P in the projectivization $\mathbb{P}\mathbb{V}$ of a 3D vector space \mathbb{V} . Lift the vertices of the polygon to vectors $V_i \in \mathbb{V}$, $i = 1, \dots, n$. Define an

operator $G_P: \mathbb{V} \rightarrow \mathbb{V}$ by the formula

$$G_P(V) := nV - \sum_{i=1}^n \frac{V_{i-1} \wedge V \wedge V_{i+1}}{V_{i-1} \wedge V_i \wedge V_{i+1}} V_i, \quad (1)$$

where all indices are understood modulo n . Note that this operator does not change under a rescaling of V_i s and hence depends only on the polygon P . What Glick proved is that the limit point of successive images of P under the pentagram map is one of the eigenvectors of G_P (equivalently, a fixed point of the associated projective mapping $\mathbb{P}\mathbb{V} \rightarrow \mathbb{P}\mathbb{V}$).

We believe that the significance of Glick's operator actually goes beyond the limit point. In particular, as was observed by Glick himself, the operator G_P has a natural geometric meaning for both pentagons and hexagons. Namely, by Clebsch's theorem, every pentagon is projectively equivalent to its pentagram map image, and it turns out that the corresponding projective transformation is given by $G_P - 3I$, where I is the identity matrix. Indeed, consider, for example, the 1st vertex of the pentagon and its lift V_1 . Then, the above formula gives

$$(G_P - 3I)(V_1) = V_1 - \frac{V_2 \wedge V_1 \wedge V_4}{V_2 \wedge V_3 \wedge V_4} V_3 - \frac{V_3 \wedge V_1 \wedge V_5}{V_3 \wedge V_4 \wedge V_5} V_4.$$

Taking the wedge product of this expression with $V_2 \wedge V_4$ or $V_3 \wedge V_5$, we get zero. This means that

$$(G_P - 3I)(V_1) \in \text{span}(V_2, V_4) \cap \text{span}(V_3, V_5),$$

so the corresponding point in the projective plane is the intersection of diagonals of the pentagon. Furthermore, since Glick's operator is invariant under cyclic permutations, the same holds for all vertices, meaning that the operator $G_P - 3I$ indeed takes a pentagon to its pentagram map image.

Likewise, the 2nd iterate of the pentagram map on hexagons also leads to an equivalent hexagon and the equivalence is again realized by $G_P - 3I$. Finally, notice that for quadrilaterals $G_P - 2I$ is a constant map onto the intersection of diagonals. These observations make us believe that the operator G_P is *per se* an important object in projective geometry, whose full significance is yet to be understood.

In the present paper, we show that Glick's operator G_P can be interpreted as *infinitesimal monodromy*. To define the latter, consider the space of *twisted polygons*, which are polygons closed up to a projective transformation, known as the *monodromy*.

Any closed polygon can be viewed as a twisted one, with trivial monodromy. To define the infinitesimal monodromy, we deform a closed polygon into a genuine twisted one. To construct such a deformation, we use what is known as the *scaling symmetry*. The scaling symmetry is a 1-parametric group of transformations of twisted polygons that commutes with the pentagram map. That symmetry was instrumental for the proof of complete integrability of the pentagram map [8].

Applying the scaling symmetry to a given closed polygon P , we get a family P_z of polygons depending on a real parameter z and such that $P_1 = P$. Thus, the monodromy M_z of P_z is a projective transformation depending on z , which is the identity for $z = 1$. By definition, the infinitesimal monodromy of P is the derivative dM_z/dz at $z = 1$. This makes the infinitesimal monodromy an element of the Lie algebra of the projective group $\mathbb{PGL}(\mathbb{P}^2)$, that is, a linear operator on \mathbb{R}^3 defined up to adding a scalar matrix. The following is our main result.

Theorem 1.1. The infinitesimal monodromy of a closed polygon P coincides with Glick's operator G_P , up to the addition of a scalar matrix.

This result provides another perspective on the limit point. Namely, observe that for $z \approx 1$ the monodromy M_z of the deformed polygon is given by

$$M_z \approx I + (z - 1)(G_P + \lambda I),$$

up to higher-order terms. Thus, the eigenvectors of G_P , and in particular the limit point, coincide with limiting positions of eigenvectors of M_z as $z \rightarrow 1$. At least one of the eigenvectors of M_z has a geometric meaning. Namely, the deformed polygon $P(z)$ can be thought of as a spiral, and the center of that spiral must be an eigenvector of the monodromy. We believe that as $z \rightarrow 1$ that eigenvector converges to the limiting point of the pentagram map (and not to one of the two other eigenvectors). If this is true, then we have the following picture. The scaling symmetry turns a closed polygon into a spiral. As the scaling parameter z goes to 1, the spiral approaches the initial polygon, while its center approaches the limit point of the pentagram map; see Figure 3.

We note that the scaling symmetry is actually only defined on projective equivalence classes of polygons as opposed to actual polygons. This makes the family of polygons P_z we used to define the infinitesimal monodromy nonunique. After reviewing the basic notions in Section 2, we show in Section 3 that the infinitesimal monodromy does not depend on the family used to define it. The proof of Theorem 1.1 is given in Section 4.

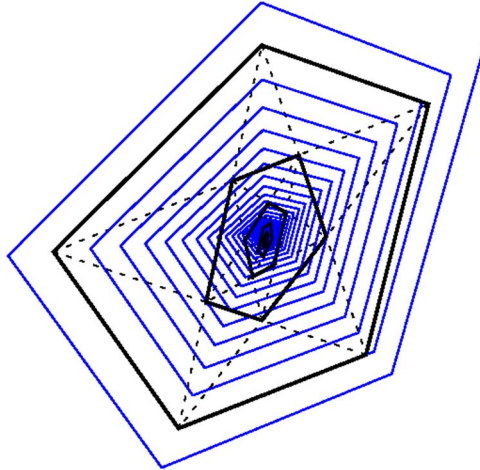


Fig. 3. The image of a closed polygon under a scaling transformation is a spiral. As the scaling parameter goes to 1, the center of the spiral approaches the limit point of the pentagram map.

We end the introduction by mentioning a possible future direction. The notion of infinitesimal monodromy is well defined for polygons in any dimension and any scaling operation. For multidimensional polygons, there are different possible scalings, corresponding to different integrable generalizations of the pentagram map [6, 7]. It would be interesting to investigate the infinitesimal monodromy in those cases, along with its possible relation to the limit point of the corresponding pentagram maps. As for now, it is not even known if such a limit point exists for any class of multidimensional polygons satisfying a convexity-type condition.

It also seems that the infinitesimal monodromy in \mathbb{P}^1 is related to the so-called cross-ratio dynamics; see [1, Section 6.2.1].

2 Background: Twisted Polygons, Corner Invariants, and Scaling

In this section, we briefly recall standard notions related to the pentagram map, concentrating on what will be used in the sequel.

A *twisted n -gon* is a bi-infinite sequence of points $v_i \in \mathbb{P}^2$ such that $v_{i+n} = M(v_i)$ for all $i \in \mathbb{Z}$ and a certain projective transformation $M \in \mathbb{PGL}(\mathbb{P}^2)$ called the *monodromy*. A twisted n -gon generalizes the notion of a closed n -gon as we recover a closed n -gon when the monodromy is equal to the identity. We denote the space of twisted n -gons by \mathcal{P}_n .

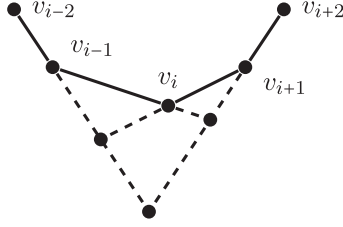


Fig. 4. Definition of corner invariants.

The pentagram map takes a twisted n -gon to a twisted n -gon (preserving the monodromy) so it can be regarded as a densely defined map from the space \mathcal{P}_n of twisted n -gons to itself. From now on, we will assume that polygons are in sufficiently general position so as to allow for all constructions to go through unhindered.

We say that two twisted n -gons $\{v_i\}$ and $\{v'_i\}$ are projectively equivalent when there is a projective transformation Φ such that $\Phi(v_i) = v'_i$. Notice that if two twisted n -gons are projectively equivalent, then their monodromies M, M' are related by $M' = \Phi \circ M \circ \Phi^{-1}$.

The pentagram map on twisted n -gons commutes with projective transformations and as such descends to a map on the space $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ of projective equivalence classes of twisted n -gons.

We now recall a construction of coordinates on the space $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ of projective equivalence classes of twisted n -gons. These coordinates are known as *corner invariants* and were introduced in [11].

Let $\{v_i \in \mathbb{P}^2\}$ be a twisted polygon. Then, the corner invariants x_i, y_i of the vertex v_i are defined as follows:

$$x_i := \left[v_{i-2}, v_{i-1}, \left((v_{i-2}, v_{i-1}) \cap (v_i, v_{i+1}) \right), \left((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}) \right) \right],$$

$$y_i := \left[\left((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}) \right), \left((v_{i-1}, v_i) \cap (v_{i+1}, v_{i+2}) \right), v_{i+1}, v_{i+2} \right],$$

where we define the cross-ratio $[a, b, c, d]$ of four points a, b, c, d on a projective line as

$$[a, b, c, d] := \frac{(a-b)(c-d)}{(a-c)(b-d)}.$$

Consider Figure 4. The value of x_i is the cross-ratio of the four points drawn on the line (v_{i-2}, v_{i-1}) (i.e., the line on the left) and y_i is the cross-ratio of the four points drawn on the line (v_{i+1}, v_{i+2}) (i.e., the line on the right).

These corner invariants are defined on almost the entire space \mathcal{P}_n of twisted n -gons. Furthermore, these numbers are invariant under projective transformations and hence descend to the space $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ of projective equivalence classes of twisted polygons. As shown in [11], the functions $x_1, \dots, x_n, y_1, \dots, y_n$ constitute a coordinate system on an open dense subset of $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$. This in particular allows one to express the pentagram map, viewed as a transformation of $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$, in terms of the corner invariants.

If we are given a twisted n -gon with corner invariants (x_i, y_i) , then the corner invariants (x'_i, y'_i) of its image under the pentagram are given by

$$x'_i = x_i \frac{1 - x_{i-1}y_{i-1}}{1 - x_{i+1}y_{i+1}} \quad y'_i = y_{i+1} \frac{1 - x_{i+2}y_{i+2}}{1 - x_i y_i}.$$

These formulas assume a specific labeling of vertices of the pentagram map image. For a different labeling, the resulting formulas differ by a shift in indices. The choice of labeling, and more generally, the specific form of the above formulas will be of no importance to us. We will only use the following corollary. Consider a 1-parametric group of densely defined transformations $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2) \rightarrow \mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ given by

$$R_z : (x_i, y_i) \mapsto (x_i z, y_i z^{-1}). \quad (2)$$

These transformations are known as *scaling symmetries*.

Proposition 2.1. The scaling symmetry $R_z : \mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2) \rightarrow \mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ on projective equivalence classes of twisted polygons commutes with the pentagram map for any $z \neq 0$.

Proof. The above formulas for the pentagram map in x, y coordinates remain unchanged if all x variables are multiplied by z and all y variables by are multiplied by z^{-1} . ■

This proposition was a key tool in the proof of integrability of the pentagram map. Namely, consider a (twisted or closed) polygon P defined up to a projective transformation, and let P_z be its image under the scaling symmetry. Then, since the pentagram map commutes with scaling and preserves the monodromy, it follows that the monodromy M_z of P_z (which does not have to be the identity even if the initial polygon is closed!) is invariant under the map. Since P_z is only defined as a projective

equivalence class, this means that M_z is only defined up to conjugation. Nevertheless, taking conjugation-invariant functions (e.g., appropriately normalized eigenvalues) of M_z , we obtain, for every z , functions that are invariant under the pentagram map. It is shown in [8] that the so-obtained functions commute under an appropriately defined Poisson bracket and turn the pentagram map into a discrete completely integrable system. See also [9] for a more detailed proof. In our paper, we utilize pretty much the same idea, but instead of looking at the eigenvalues of M_z , we will consider M_z itself. It is not quite well defined, but we will show that its z derivative at $z = 1$ is and that it coincides with Glick's operator.

3 Infinitesimal Monodromy

In this section, we define the infinitesimal monodromy and show that it does not depend on the choices we need to make to formulate the definition, namely on the way we lift the scaling symmetry (2) from projective equivalence classes of polygons to actual polygons.

We start with a closed n -gon, P , in \mathbb{P}^2 . Let $[P] \in \mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ be its projective equivalence class. Then, applying the scaling transformation R_z given by (2) to $[P]$, we get a path $R_z[P]$ in $\mathcal{P}_n / \mathbb{PGL}(\mathbb{P}^2)$ such that $R_1[P] = [P]$. Now, choose a smooth in z lift P_z of the path $R_z[P]$ to the space \mathcal{P}_n of actual twisted polygons such that $P_1 = P$ (we will construct an explicit example of such a lift later on). Denote by $M_z \in \mathbb{PGL}(\mathbb{P}^2)$ the monodromy of P_z . It is a family of projective transformations such that M_1 is the identity, $M_1 = I$. This family does depend on the choice of the lift P_z of the path $R_z[P]$. However, as we show below, the tangent vector dM_z/dz at $z = 1$ does not depend on that choice and this is what we call the *infinitesimal monodromy*.

Definition 3.1. The *infinitesimal monodromy* of a closed polygon P is the derivative dM_z/dz at $z = 1$, where M_z is the monodromy of any path P_z of polygons such that $P_1 = P$ and $[P_z] = R_z[P]$.

The infinitesimal monodromy is therefore a tangent vector to the projective group $\mathbb{PGL}(\mathbb{P}^2)$ at the identity, and, upon a choice of basis, can be viewed as a 3×3 matrix defined up to addition of a scalar matrix. Our main result can thus be formulated as follows.

Theorem 3.1 (=Theorem 1.1). The tangent vector to $\mathbb{PGL}(\mathbb{P}^2)$ represented by Glick's operator G_P coincides with the infinitesimal monodromy of P .

The proof will be given in Section 4. But first, we need to check that Definition 3.1 makes sense, that is, that the infinitesimal monodromy does not depend on the choice of the path P_z . This is established by the following.

Proposition 3.2. Let P_z and \tilde{P}_z be two families of polygons such that $P_1 = \tilde{P}_1$ is a closed polygon and \tilde{P}_z is projectively equivalent to P_z for every z . Then, for the monodromies M_z and \tilde{M}_z of these families, at $z = 1$, we have $dM_z/dz = d\tilde{M}_z/dz$.

Proof. Let Φ_z be a projective transformation taking P_z to \tilde{P}_z . Since $P_1 = \tilde{P}_1$, we have that $\Phi_1 = I$ (a generic n -gon in \mathbb{P}^2 does not admit any nontrivial projective automorphisms, provided that $n \geq 4$). Then, we know that the monodromies are related by $\tilde{M}_z = \Phi_z M_z \Phi_z^{-1}$. Differentiating this and using that $\Phi_1 = I$, we get

$$\frac{d}{dz} \Big|_{z=1} \tilde{M}_z = \frac{d}{dz} \Big|_{z=1} M_z + \left[\frac{d}{dz} \Big|_{z=1} \Phi_z, M_1 \right].$$

This identity in particular shows that the infinitesimal monodromy of a *twisted* polygon is in general not well defined, due to the extra commutator term in the right-hand side. But for a closed polygon, we have $M_1 = I$, so the extra term vanishes and we get the desired identity. ■

Before we proceed to the proof of the main theorem, let us mention one property of the infinitesimal monodromy.

Proposition 3.3. The infinitesimal monodromy of a closed polygon is preserved by the pentagram map.

Proof. The pentagram map preserves the monodromy and commutes with the scaling. The infinitesimal monodromy is defined using monodromy and scaling and is thus preserved as well. ■

This result in fact follows from our main theorem because Glick shows in [5, Theorem 3.1] that his operator has this property. However, the proof based on Glick's definition is quite nontrivial, while in our approach, it is immediate. The observation that the infinitesimal monodromy is preserved by the pentagram map was in fact our motivation to conjecture that it should coincide with Glick's operator. And, as we show below, this is indeed true.

4 The Infinitesimal Monodromy and Glick's Operator

In this section, we prove our main result, Theorem 1.1 (=Theorem 3.1). To that end, we explicitly construct a deformation P_z of a polygon P as in Definition 3.1. Such a deformation is not unique, but we know that the infinitesimal monodromy does not depend on the deformation. We will in fact use this ambiguity to our advantage by choosing a deformation for which the infinitesimal monodromy can be computed explicitly. We will then compute it and see that it coincides with Glick's operator.

Consider a closed n -gon P . Lift the n -periodic sequence $\{v_i \in \mathbb{P}^2\}$ of its vertices to an n -periodic sequence of nonzero vectors $V_i \in \mathbb{R}^3$. Then, for every $i \in \mathbb{Z}$, there exist $a_i, b_i, c_i \in \mathbb{R}$ such that

$$V_{i+3} = a_i V_{i+2} + b_i V_{i+1} + c_i V_i. \quad (3)$$

Furthermore, for a generic polygon, the numbers a_i, b_i, c_i are uniquely determined because the points v_i, v_{i+1}, v_{i+2} are not collinear so the vectors V_i, V_{i+1}, V_{i+2} are linearly independent. Also, we have $c_i \neq 0$ for any i because the points $v_{i+1}, v_{i+2}, v_{i+3}$ are not collinear. In addition to that, since $V_{i+n} = V_i$ we have that the sequences a_i, b_i, c_i are n -periodic. Finally, notice that for fixed a_i, b_i, c_i the sequence V_i is uniquely determined by equation (3) and initial condition V_0, V_1, V_2 . Indeed, given V_0, V_1, V_2 and using that $c_i \neq 0$, we can successively find all V_i s from (3). This gives us a way to deform the polygon P : keeping V_0, V_1, V_2 unchanged, we deform the coefficients in (3). Namely, consider the following equation:

$$V_{i+3} = a_i V_{i+2} + z^{-1}(b_i V_{i+1} + c_i V_i). \quad (4)$$

We assume that the vectors V_0, V_1, V_2 do not depend on z and coincide with the above-constructed lifts of vertices of P . For any $z \neq 0$, equation (4) has a unique solution with such initial condition. For $z = 1$, we recover the initial polygon, while for other values of z , we get its deformation. Note that for $i \neq 0, 1, 2$, the solutions V_i of (4) are actually functions of the parameter z , that is, $V_i = V_i(z)$.

Proposition 4.1. Taking the solution of (4) such that V_0, V_1, V_2 are fixed lifts of vertices v_0, v_1, v_2 of P and projecting the vectors $V_i \in \mathbb{R}^3$ to \mathbb{P}^2 , we get a family P_z of twisted polygons as in Definition 3.1. Namely, we have that $P_1 = P$, and also $[P_z] = R_z[P]$, where R_z is the scaling symmetry (2).

Proof. First, note that if a sequence V_i is a solution of (4) with given initial condition, then $V_i(z) \neq 0$ for any i and every z sufficiently close to 1, so we can indeed project those vectors to get a sequence of points in \mathbb{P}^2 . Indeed, for $z = 1$, this is so by construction and hence is also true for nearby values of z by continuity (in fact, one can show that $V_i(z) \neq 0$ for any $z \neq 0$, not necessarily close to 1).

Further, observe that since the coefficients of equation (4) are periodic, its solution is quasi-periodic: $V_{i+n}(z) = M_z V_i(z)$ for a certain invertible matrix M_z depending on z . Therefore, the projections $v_i(z) \in \mathbb{P}^2$ of the vectors $V_i(z) \in \mathbb{R}^3$ form a twisted polygon whose monodromy is the projective transformation defined by M_z . Furthermore, since equations (3) and (4) agree for $z = 1$, and the initial conditions are the same, too, it follows that for the so-obtained family P_z of twisted polygons, we have $P_1 = P$. Finally, we need to show that the projective equivalence classes of P and P_z are related by scaling $[P_z] = R_z[P]$. To that end, we use formulas expressing corner invariants in terms of coefficients of a recurrence relation satisfied by the lifts of vertices. Arguing as in the proof of [8, Lemma 4.5], one gets the following expressions for the corner invariants of P :

$$x_{i+2} = \frac{a_i c_i}{b_i b_{i+1}} \quad Y_{i+2} = -\frac{b_{i+1}}{a_i a_{i+1}}.$$

Accordingly, since equations (3) and (4) encoding P and P_z are connected by the transformation $b_i \mapsto z^{-1}b_i$, $c_i \mapsto z^{-1}c_i$, the corner invariants of P_z are given by

$$x_{i+2}(z) = \frac{a_i(z^{-1}c_i)}{(z^{-1}b_i)(z^{-1}b_{i+1})} = z x_{i+2} \quad Y_{i+2}(z) = -\frac{z^{-1}b_{i+1}}{a_i a_{i+1}} = z^{-1} Y_{i+2}.$$

Thus, the projective equivalence classes of the polygons P and P_z are indeed related by scaling, as desired. ■

We are now in a position to prove our main result. To that end, we will compute the monodromy of the polygon defined by (4), take its derivative at $z = 1$, and hence find the infinitesimal monodromy.

We put the vectors $V_i(z)$ into columns of matrices as follows: define

$$W_i(z) := [V_{i+2}(z) \quad V_{i+1}(z) \quad V_i(z)].$$

Then, the relation (4) gives us the matrix equation

$$W_{i+1}(z) = W_i(z)U_i(z),$$

where

$$U_i(z) := \begin{bmatrix} a_i & 1 & 0 \\ z^{-1}b_i & 0 & 1 \\ z^{-1}c_i & 0 & 0 \end{bmatrix}. \quad (5)$$

We stop explicitly recording the dependence on z as it is notationally cumbersome. Inductively, we have that

$$W_i = W_0 U_0 U_1 \dots U_{i-1}.$$

In particular,

$$W_n = W_0 U,$$

where $U := U_0 U_1 \dots U_{n-1}$. At the same time, we have that $V_{i+n} = M_z V_i$, where M_z is a matrix representing the monodromy of the polygon defined by the vectors V_i . This means that $W_n = M_z W_0$. Relating these two expressions for W_n , we get

$$W_0 U = M_z W_0 \iff M_z = W_0 U W_0^{-1}.$$

Notice that because V_0, V_1, V_2 are fixed, we have that $W_0 = [V_0 \ V_1 \ V_2]$ is constant while z varies. This means that all the dependence of M_z on z is contained in the expression for U . This gives

$$\begin{aligned} \frac{dM_z}{dz} &= \frac{d}{dz} (W_0 U_0 \dots U_{n-1} W_0^{-1}) \\ &= \sum_{i=0}^{n-1} W_0 U_0 \dots U_{i-1} \frac{dU_i}{dz} U_{i+1} \dots U_{n-1} W_0^{-1} = \sum_{i=0}^{n-1} W_i \frac{dU_i}{dz} U_{i+1} \dots U_{n-1} W_0^{-1}, \end{aligned}$$

where the last equality uses that $W_i = W_0 U_0 \dots U_{i-1}$. Further, observe that

$$U_{i+1} \dots U_{n-1} = (U_0 \dots U_i)^{-1} (U_0 \dots U_{n-1}) = (W_0^{-1} W_{i+1})^{-1} (W_0^{-1} W_n) = W_{i+1}^{-1} W_n.$$

Also, using that $W_n W_0^{-1} = M_z$, we get

$$\frac{dM_z}{dz} = \sum_{i=0}^{n-1} W_i \frac{dU_i}{dz} W_{i+1}^{-1} W_n W_0^{-1} = \left(\sum_{i=0}^{n-1} W_i \frac{dU_i}{dz} W_{i+1}^{-1} \right) M_z.$$

Further, using that the monodromy satisfies $M_1 = I$ because we started with a closed n -gon, we arrive at

$$\frac{dM_z}{dz} \Big|_{z=1} = \sum_{i=0}^{n-1} S_i,$$

where

$$S_i := \left(W_i \frac{dU_i}{dz} W_{i+1}^{-1} \right) \Big|_{z=1}.$$

Now, we will show that summing these S_i with $i = 0, 1, \dots, n-1$ gives (1) up to a scalar matrix. Using (5), we get

$$\frac{dU_i}{dz} \Big|_{z=1} = \begin{bmatrix} 0 & 0 & 0 \\ -b_i & 0 & 0 \\ -c_i & 0 & 0 \end{bmatrix}.$$

Further, observe that for $z = 1$ the matrix W_i sends the standard basis to the lifts V_{i+2}, V_{i+1}, V_i of the vertices of P . Therefore, W_{i+1}^{-1} takes the vectors $V_{i+3}, V_{i+2}, V_{i+1}$ to the standard basis, from which we find that the matrix S_i acts on these vectors as

$$V_{i+3} \mapsto -b_i V_{i+1} - c_i V_i, \quad V_{i+2} \mapsto 0 \quad V_{i+1} \mapsto 0.$$

Using also (3), we find that

$$S_i(V_i) = \frac{1}{c_i} S_i(V_{i+3}) = -\frac{b_i}{c_i} V_{i+1} - V_i,$$

which means that

$$S_i(V) = \frac{|V, V_{i+1}, V_{i+2}|}{|V_i, V_{i+1}, V_{i+2}|} \left(-V_i - \frac{b_i}{c_i} V_{i+1} \right) \quad \forall V \in \mathbb{R}^3,$$

where $|A, B, C|$ is the determinant of the matrix with columns A, B, C . Further, rewriting (3) as

$$-V_i - \frac{b_i}{c_i} V_{i+1} = \frac{a_i}{c_i} V_{i+2} - \frac{1}{c_i} V_{i+3},$$

we get

$$S_i(V) = \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_i|} \left(\frac{a_i}{c_i} V_{i+2} - \frac{1}{c_i} V_{i+3} \right) = \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, c_i^{-1} V_{i+3}|} \left(\frac{a_i}{c_i} V_{i+2} - \frac{1}{c_i} V_{i+3} \right),$$

where in the last equality, we used (3) to express V_i in terms of $V_{i+1}, V_{i+2}, V_{i+3}$. This can be rewritten as

$$S_i(V) = \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} a_i V_{i+2} - \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+3}, \quad (6)$$

and the 1st term can be further rewritten as

$$\begin{aligned} \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} a_i V_{i+2} &= \frac{|V_{i+1}, a_i V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+2} = \frac{|V_{i+1}, V_{i+3} - c_i V_i, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+2} \\ &= -\frac{|V_{i+1}, V, V_{i+3}|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+2} + \frac{|V_i, V_{i+1}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} c_i V_{i+2}, \end{aligned} \quad (7)$$

where in the 2nd equality, we used (3) to express $a_i V_{i+2}$ in terms of V_i, V_{i+1}, V_{i+3} . Furthermore, using (3) to express V_{i+3} in terms of V_i, V_{i+1}, V_{i+2} , the last term in the latter expression can be rewritten as

$$\frac{|V_i, V_{i+1}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} c_i V_{i+2} = \frac{|V_i, V_{i+1}, V|}{|V_i, V_{i+1}, V_{i+2}|} V_{i+2}. \quad (8)$$

Combining (6), (7), and (8), we arrive at the following expression:

$$S_i(V) = -\frac{|V_{i+1}, V, V_{i+3}|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+2} + \frac{|V_i, V_{i+1}, V|}{|V_i, V_{i+1}, V_{i+2}|} V_{i+2} - \frac{|V_{i+1}, V_{i+2}, V|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+3}.$$

Since the last two terms only differ by a shift in index, and the sequence of V_i s in n -periodic, we get

$$\frac{dM_z}{dz} \Big|_{z=1}(V) = -\sum_{i=0}^{n-1} S_i(V) = \sum_{i=0}^{n-1} \frac{|V_{i+1}, V, V_{i+3}|}{|V_{i+1}, V_{i+2}, V_{i+3}|} V_{i+2} = -\sum_{i=0}^{n-1} \frac{|V_{i-1}, V, V_{i+1}|}{|V_{i-1}, V_i, V_{i+1}|} V_i,$$

which coincides with Glick's operator (1) up to a scalar matrix. Thus, Theorem 1.1 (=Theorem 3.1) is proved.

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