Stability in bihamiltonian systems and multidimensional rigid body

Anton Izosimov

Department of Mechanics and Mathematics, Moscow State University, Leninskie Gori, 1, 119991, Moscow, Russia

ARTICLE INFO

ABSTRACT

The presence of two compatible hamiltonian structures is known to be one of the main, and the most natural, mechanisms of integrability. For every pair of hamiltonian structures, there are associated conservation laws (first integrals). Another approach is to consider the second hamiltonian structure on its own as a tensor conservation law. The latter is more intrinsic as compared to scalar conservation laws derived from it and, as a rule, it is "simpler". Thus it is natural to ask: can the dynamics of a bihamiltonian system be understood by studying its hamiltonian pair, without studying the associated first integrals?

In this paper, the problem of stability of equilibria in bihamiltonian systems is considered and it is shown that the conditions for nonlinear stability can be expressed in algebraic terms of linearization of the underlying Poisson pencil. This is used to study stability of stationary rotations of a free multidimensional rigid body.

1. Introduction

Since the pioneering works [1–3], the presence of two compatible Hamiltonian structures is known to be one of the main, and the most natural, mechanisms of integrability. This mechanism is responsible for the integrability of many equations coming from mechanics, mathematical physics and geometry (see, for example, [4] and references therein). The idea is that for every pair of Hamiltonian structures, there are associated conservation laws (first integrals).

Accordingly, a biHamiltonian structure is usually considered as a "factory of conservation laws". However, the second Hamiltonian structure on its own can be considered as a tensor conservation law. The latter is more intrinsic as compared to scalar conservation laws derived from it and, as a rule, it is "simpler". For example, the second Poisson structure for the Korteweg–de Vries equation [1] is linear, while the first integrals are complicated polynomials given by a recurrence formula. Thus it is natural to ask: can the dynamics of a biHamiltonian system be understood by studying its Hamiltonian pair, without studying the associated first integrals?

In this paper, the problem of stability of equilibria in biHamiltonian systems is considered and it is shown that the conditions for *nonlinear* stability in the biHamiltonian case can be expressed in terms of the *linear* part of the underlying Poisson pencil. This linear part appears to be a collection of Lie algebras, each carrying a two-cocycle. Thus, the problem of stability in biHamiltonian systems can be considered as algebraic.

As it was noted above, the notions "biHamiltonian" and "integrable" are closely related. For this reason, the method discussed in this paper can be viewed as a rather general method for stability investigation in integrable systems. This paper focuses on the finite-dimensional case, however an essential part of the construction works in infinite dimension as well.[1] Also note that the theorem formulated in this paper can be easily generalized from equilibria to periodic and general quasi-periodic trajectories.

E-mail address: izosimov@mech.math.msu.su.

[1] Certainly, the infinite-dimensional case needs a separate discussion.

To the author's knowledge, the idea of studying dynamics by means of a biHamiltonian structure was first suggested by Bolsinov. In his paper [5] biHamiltonian structure is used to describe the singular set of an integrable Hamiltonian system. Further developments are presented in the papers [6,7] devoted to a more detailed analysis of singularities of biHamiltonian systems. In particular, in [7] the notion of linearization of a Poisson pencil is introduced. This notion is used in the present paper to express stability conditions.

As an application, the stability problem for stationary rotations of a free multidimensional rigid body is solved. On the one hand, this problem is too complicated to be solved by a direct method (such as the Arnold method, see below), because the first integrals are polynomials of high degree. On the other hand, the biHamiltonian structure of this problem is simple enough (in other words, the problem has complicated scalar conservation laws, but simple tensor conservation laws). This circumstance makes the application of the biHamiltonian approach to a multidimensional rigid body extremely effective and provides a simple method for the determination of its stability.

2. The Arnold method

In the Hamiltonian case, stability in a linear approximation is always neutral and thus insufficient for a conclusion about nonlinear stability. To prove nonlinear stability in a Hamiltonian system, one usually uses the Arnold method (also known as the Energy-Casimir method, see [8]). The Arnold method can be formulated as follows:

Theorem 1. *Consider a Hamiltonian system v on a Poisson manifold. Let x be an equilibrium point of v which belongs to a generic symplectic leaf O. Then x is a critical point for the restriction of the energy to O. If this critical point is a non-degenerate minimum or maximum, then x is stable.*

Since we deal with the finite-dimensional case, this method proves the nonlinear (Lyapunov) stability.

Now let the system under consideration be integrable. Then one can replace the energy in the formulation of the Arnold method by any linear combination of the conserved quantities. The extended method can be formulated as follows:

Theorem 2. *Consider an integrable Hamiltonian system v on a Poisson manifold. Let x be an equilibrium point of v which belongs to a generic symplectic leaf O. Let also $f_1, \ldots, f_n$ be the first integrals of the system. Suppose that there exists a linear combination $f = \sum a_i f_i$ such that $\mathrm{d}f|_O = 0$ and $\mathrm{d}^2 f|_O > 0$. Then x is stable.*

This extended formulation of the Arnold method is a powerful tool for investigating stability in integrable Hamiltonian systems. Nevertheless, in many dimensions it may be very complicated to compute the second differentials $\mathrm{d}^2 f_i$ and find a linear combination of them satisfying the conditions of the theorem. However, it turns out, that in the biHamiltonian case this calculation can be replaced by a verification of a certain algebraic condition.

3. Definitions

3.1. Poisson pencils and biHamiltonian vector fields

Definition 1. Two Poisson brackets on a manifold M are called *compatible*, if any linear combination of them is a Poisson bracket again. The *Poisson pencil* generated by two compatible Poisson brackets P_0, P_∞ is the set

$$\Pi = \{P_\lambda = P_0 - \lambda P_\infty\}_{\lambda \in \overline{\mathbb{R}}}. \tag{1}$$

Sometimes it also makes sense to consider $\lambda \in \overline{\mathbb{C}}$.

Remark 1. A Poisson pencil can also be defined as the set of all non-trivial linear combinations of two compatible brackets. However, it makes sense to consider Poisson brackets only up to proportionality, thus these two definitions may be considered as equivalent.

The minus sign in (1) is conventional.

Example 1. Let $\mathfrak{g}$ be an arbitrary Lie algebra, $a \in \mathfrak{g}^*$. Consider the Lie–Poisson bracket[2] given by $P_0(x)(\xi, \eta) = x([\xi, \eta])$ and a constant bracket given by $P_\infty(x)(\xi, \eta) = a([\xi, \eta])$, the so-called bracket "with the frozen argument". It is easy to see that P_0 and P_∞ are compatible.

These two brackets are related to the so-called "argument shift method" introduced by Mishchenko and Fomenko [9].

Definition 2. A vector field is biHamiltonian with respect to a given pencil if it is Hamiltonian with respect to *all* brackets of the pencil.

---

[2] Throughout the whole paper Poisson brackets are identified with their Poisson tensors.

3.2. Rank and spectrum of a Poisson pencil

Definition 3. The *rank of a pencil Π at a point x* is the number

$$\operatorname{rank} \Pi(x) = \max_{\lambda \in \overline{\mathbb{C}}} \operatorname{rank} P_\lambda(x). \tag{2}$$

The *rank of a pencil Π* (on a manifold M) is the number

$$\operatorname{rank} \Pi = \max_{x \in M} \operatorname{rank} \Pi(x). \tag{3}$$

Definition 4. The *spectrum* of a pencil Π at a point x is the set

$$\Lambda_\Pi(x) = \{\lambda \in \overline{\mathbb{C}} \mid \operatorname{rank} P_\lambda(x) < \operatorname{rank} \Pi(x)\}. \tag{4}$$

Example 2. Let Π be the pencil from Example 1. If a is regular, then the spectrum $\Lambda_\Pi(x)$ consists of $\lambda \in \mathbb{C}$ such that $x - \lambda a$ is singular in $\mathbb{C} \otimes \mathfrak{g}^*$. If a is singular, then the spectrum additionally contains $\lambda = \infty$.

3.3. Linear Poisson pencils

Definition 5. Let $\mathfrak{g}$ be a Lie algebra and A be a skew-symmetric bilinear form on it. Then A can be considered as a Poisson tensor on the dual space $\mathfrak{g}^*$. Assume that the corresponding bracket is compatible with the Lie–Poisson bracket. In this case the Poisson pencil $\Pi(\mathfrak{g}, A)$ generated by these two brackets is called the *linear pencil* associated with the pair $(\mathfrak{g}, A)$.

Example 3. The pencil from Example 1 is linear.

The following is well known.

Proposition 1. *A form A on $\mathfrak{g}$ is compatible with the Lie–Poisson bracket if and only if this form is a Lie algebra 2-cocycle, i.e.*

$$dA(\xi, \eta, \zeta) = A([\xi, \eta], \zeta) + A([\eta, \zeta], \xi) + A([\zeta, \xi], \eta) = 0 \tag{5}$$

for any $\xi, \eta, \zeta \in \mathfrak{g}$.

3.4. Linearization of a Poisson pencil

Let P be a Poisson bracket. It is well-known that the linear part of P at a point x defines a natural Lie algebra structure on $\operatorname{Ker} P(x)$. This Lie algebra is called the linearization of P at x. Now consider a Poisson pencil $\Pi = \{P_\lambda\}$ and fix a point x. Denote by $\mathfrak{g}_\lambda(x)$ the linearization of P_λ at the point x.

It turns out that apart from the Lie algebra structure $\mathfrak{g}_\lambda$ carries one more additional structure.

Proposition 2. 1. *For any α and β the restrictions of $P_\alpha(x)$, $P_\beta(x)$ on $\mathfrak{g}_\lambda(x)$ coincide up to a multiplicative constant.*
2. *The 2-form $P_\alpha|_{\mathfrak{g}_\lambda}$ is a 2-cocycle on $\mathfrak{g}_\lambda$.*

Consequently, $P_\alpha|_{\mathfrak{g}_\lambda}$ defines a linear Poisson pencil on $\mathfrak{g}_\lambda^*$. Since $P_\alpha|_{\mathfrak{g}_\lambda}$ is defined up to a multiplicative constant, the pencil is well-defined. Denote this pencil by $d_\lambda \Pi(x)$.

Definition 6. The pencil $d_\lambda \Pi(x)$ is called the *λ-linearization* of the pencil Π at x.

The linearization of a Poisson pencil at a given point is, therefore, not a single pencil, but a whole "curve" of linear Poisson pencils parametrized by $\lambda \in \overline{\mathbb{C}}$. However, if $\operatorname{rank} \Pi(x) = \operatorname{rank} \Pi$, then it is easy to see that $d_\lambda \Pi(x)$ is non-trivial only for $\lambda \in \Lambda_\Pi(x)$.

Example 4. Consider the pencil from Example 1. The algebra $\mathfrak{g}_\lambda(x)$ in this case is simply the stabilizer of $x - \lambda a$. The second form $P_\alpha|_{\mathfrak{g}_\lambda}$ is given on the stabilizer by the formula $a|_{\mathfrak{g}_\lambda}([\xi, \eta])$. Thus, the λ-linearization is the "restriction" of the initial pencil to the stabilizer of $x - \lambda a$. If λ is not in the spectrum, then this stabilizer is abelian, and the linearization is trivial.

Remark 2. Note that it is natural to expect that a "linearization" of an object defined on a manifold M is an object defined on the tangent space $T_x M$. For a λ-linearization of a Poisson pencil this is not so: it is defined on $(\operatorname{Ker} P_\lambda(x))^*$. However, the natural inclusion map $\operatorname{Ker} P_\lambda(x) \to T_x^* M$ induces an isomorphism

$$T_x M / T_x O_\lambda(x) \simeq (\operatorname{Ker} P_\lambda(x))^*, \tag{6}$$

where $O_\lambda(x)$ is the symplectic leaf of P_λ passing through x. Thus, $d_\lambda \Pi(x)$ can be considered as a Poisson pencil on the quotient $T_x M / T_x O_\lambda(x)$.

3.5. Compact linear pencils

Let A be a 2-cocycle on a Lie algebra $\mathfrak{g}$. For an arbitrary element $\nu \in \operatorname{Ker} A$ define the bilinear form $A^{\nu}(\xi, \eta) = A([\nu, \xi], \eta)$. The cocycle identity (5) implies that this form is symmetric. Furthermore, $\operatorname{Ker} A^{\nu} \supset \operatorname{Ker} A$, therefore A^{ν} is a well-defined symmetric form on $\mathfrak{g}/\operatorname{Ker} A$.

Definition 7. A linear pencil $\Pi(\mathfrak{g}, A)$ is *compact* if there exists $\nu \in \mathcal{Z}(\operatorname{Ker} A)$ such that A^{ν} is positive-definite on $\mathfrak{g}/\operatorname{Ker} A$.

Remark 3. $\mathcal{Z}$ stands for the center of a Lie algebra.

Example 5. Any linear pencil on a compact semisimple Lie algebra is compact. Indeed, let $\mathfrak{g}$ be a compact semisimple Lie algebra. Since $H^2(\mathfrak{g}) = 0$, any cocycle A on $\mathfrak{g}$ has the form $A(\xi, \eta) = \langle a, [\xi, \eta] \rangle$, where $\langle \, , \rangle$ is the Killing form. It is easy to see that for $\nu = a$ the form A^{ν} is positive-definite on $\mathfrak{g}/\operatorname{Ker} A$.

Example 6. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Again, any cocycle on $\mathfrak{g}$ has the form $A(\xi, \eta) = \langle a, [\xi, \eta] \rangle$. Suppose that $a \neq 0$. Then it is easy to see that $\Pi(\mathfrak{g}, A)$ is compact if and only if the Killing form is negative on a. A suitable choice of ν is $\nu = -a$.

Example 7. Let $\mathfrak{g} = \operatorname{Vect}(S^1)$ be a Lie algebra of vector fields on a circle and A be the Gelfand–Fuks cocycle (see [4]):

$$A(\phi, \psi) = \int_0^{2\pi} \phi \psi''' \mathrm{d}x. \tag{7}$$

Then the pencil $\Pi(\mathfrak{g}, A)$ is compact. Indeed, if we choose $\nu = 1$, then

$$A^{\nu}(\phi, \phi) = \int_0^{2\pi} (\phi'')^2 \mathrm{d}x. \tag{8}$$

3.6. Geometric meaning of compactness condition

Proposition 3. *Suppose that a system v is biHamiltonian with respect to a compact linear pencil. Then the origin is a stable equilibrium of v.*

The idea of the proof is that the form $A^{\nu} > 0$ can be used to construct a positive-definite integral of v.

A similar statement is true for nonlinear pencils, see Theorem 4. In this case stability can be studied by checking the compactness of the *linearizations*. In contrast to the classical linearization procedure, which can only prove linearized stability, biHamiltonian linearization (defined above) proves *nonlinear* stability.

3.7. Diagonalizability condition

Definition 8. The pencil Π is called diagonalizable at x if

$$\dim \operatorname{Ker} \left(P_{\alpha}(x)|_{P_{\lambda}(x)} \right) = \operatorname{corank} \Pi(x) \quad \text{for all } \lambda \in \Lambda_{\Pi}(x), \ \alpha \neq \lambda. \tag{9}$$

Remark 4. The Jordan–Kronecker theorem (see [10]) claims that two skew-symmetric forms on a vector space can be simultaneously brought to a certain block-diagonal form. This form contains blocks of two types: Jordan blocks and Kronecker blocks. The diagonalizability condition means that all Jordan blocks for $P_0(x)$, $P_{\infty}(x)$ have size 2×2.

Example 8. Let Π be the pencil from Example 1. Suppose that a is regular. Then the pencil is diagonalizable at x if for each $\lambda \in \Lambda_{\Pi}(x)$ the following two conditions hold:

1. The index[3] of the stabilizer of $x - \lambda a$ equals the index of $\mathfrak{g}$.
2. The restriction of a to the stabilizer of $x - \lambda a$ is a regular element.

---

[3] Recall that the index of a Lie algebra $\mathfrak{g}$ can be defined as the corank of the corresponding Lie–Poisson structure, or, equivalently, as the dimension of the stabilizer of a regular element $a \in \mathfrak{g}^*$.

3.8. Regularity condition

Let v be a system which is biHamiltonian with respect to Π, $v(x) = 0$. Suppose that rank $\Pi(x) = $ rank Π.

Definition 9. Say that x is *regular* if the following condition holds:

$$\operatorname{Ker} P_\alpha(x) = \operatorname{Ker} P_\beta(x) \quad \text{for all } \alpha, \beta \notin \Lambda(x). \tag{10}$$

Theorem 3 (*Bolsinov and Oshemkov [6]*). *Let v be a system which is biHamiltonian with respect to Π, $v(x) = 0$. Suppose that* rank $\Pi(x) = $ rank Π. *Then, if x is not regular, we can find an integral f of v and $\alpha \in \mathbb{R}$ such that $P_\alpha \mathrm{d} f(x) \neq 0$.*

Consequently, if x is not regular, the whole trajectory of $P_\alpha \mathrm{d} f$ passing through x consists of equilibrium points of v. In this situation it can be shown that, provided the system is non-resonant,[4] x cannot be Lyapunov stable. Therefore, it only makes sense to study regular equilibria for stability.

4. Stability theorem

Theorem 4 (*Stability Theorem*). *Suppose that Π is a Poisson pencil on a finite-dimensional manifold, v is biHamiltonian with respect to Π. Let x be an equilibrium of v. Assume that*

1. rank $\Pi(x) = $ rank Π.
2. *The equilibrium x is regular.*
3. *The pencil Π is diagonalizable at x.*
4. *For each $\lambda \in \Lambda_\Pi(x)$ the λ-linearization $\mathrm{d}_\lambda \Pi(x)$ is compact.*

Then x is Lyapunov (nonlinearly) stable.

The proof is given in Section 6.

Remark 5. Condition 4 implies that the spectrum of Π at x is real, since a pencil on a complex Lie algebra cannot be compact.

Theorem 4 is a biHamiltonian reformulation of Theorem 2 in the following sense. Let a system v be biHamiltonian with respect to a pencil Π. Then the Casimir functions of all brackets of the pencil are first integrals of v. These first integrals are known to be in involution (see [3]). Consider the family $\mathcal{F}$ generated by all these first integrals. Then the following is true: *if the first condition of* Theorem 4 *is satisfied, then the subsequent conditions are equivalent to the existence of $f \in \mathcal{F}$ satisfying the conditions of* Theorem 2.

If $\mathcal{F}$ happens to exhaust all the first integrals of v, then Theorems 2 and 4 are equivalent (for generic points satisfying rank $\Pi(x) = $ rank Π). This should be expected if the first integrals belonging to $\mathcal{F}$ are sufficient for complete Liouville integrability of v.

Definition 10. A pencil is called *Kronecker* if its spectrum is empty almost everywhere.

Remark 6. This condition means that the Jordan–Kronecker normal form (see Remark 4) for $P_0(x)$, $P_\infty(x)$ contains only Kronecker blocks for almost all x.

Theorem 5 (*Bolsinov [5]*). *Let v be biHamiltonian with respect to a pencil Π and $\mathcal{F}$ be the family of first integrals of v described above. Then the first integrals belonging to $\mathcal{F}$ are sufficient for complete Liouville integrability of v if and only if Π is Kronecker.*

So, Theorem 4 should be the most effective for Kronecker pencils.

5. Multidimensional rigid body

5.1. Statement of the problem

It is well known that a free asymmetric three-dimensional rigid body admits three stationary rotations.[5] These are the rotations around three principal axes of inertia. The rotations around the long and the short axes are stable, while the rotation around the intermediate axis is unstable (see [8]). The problem is to obtain a multidimensional generalization of this fact, i.e. to study stationary rotations of a free multidimensional rigid body for stability.

This problem has been studied by many people, see [12–17]. However, no general solution is known.

---

[4] Recall that an integrable system is called non-resonant if its trajectories are dense on almost all Liouville tori. See [11].

[5] A rotation is called stationary if the axis of rotation is time independent.

5.2. The Euler–Arnold equations and the biHamiltonian structure

The dynamics of the angular velocity matrix Ω of a free multidimensional rigid body is described by the Euler–Arnold equations (see [8])

$$\dot{\Omega}J + J\dot{\Omega} = [J, \Omega^2], \tag{11}$$

where J is the mass tensor (see below).

Remark 7. The proof of integrability of (11) belongs to Manakov [18]. For this reason the system (11) is also known as the Manakov top.

The following two observations allow the application of Theorem 4 to the problem of stability of stationary rotations:

1. Stationary rotations are just the equilibria of (11).
2. The system (11) is biHamiltonian (with respect to a Kronecker pencil), as it was observed by Bolsinov [5]. The first Poisson structure (due to Arnold) is the standard Lie–Poisson structure on $\mathfrak{so}(n)^*$. The second (due to Bolsinov) is also a Lie–Poisson structure, but for a non-standard commutator on $\mathfrak{so}(n)$ given by $[X, Y] = XJ^2Y - YJ^2X$.

5.3. Rotation of a multidimensional body

First, consider how an n-dimensional body may rotate. At each moment of time $\mathbb{R}^n$ is decomposed into a sum of m pairwise orthogonal two-dimensional planes $\Pi_1, \ldots, \Pi_m$ and a space Π_0 of dimension $n - 2m$ orthogonal to all these planes:

$$\mathbb{R}^n = \left(\bigoplus_{i=1}^m \Pi_i\right) \oplus \Pi_0. \tag{12}$$

There is an independent rotation in each of the planes $\Pi_1, \ldots, \Pi_m$, while Π_0 is fixed.[6] In other words, a rotation of a multidimensional body can be represented as a superposition of "elementary" 2-dimensional rotations.

A rotation is stationary if all the planes $\Pi_0, \ldots, \Pi_m$ are time independent (this condition automatically implies that the velocities of the rotations are also constant).

Before studying stationary rotations for stability it is necessary to find these rotations. Recall that a rotation of a generic three-dimensional rigid body is stationary if and only if it is a rotation around one of the principal axes of inertia. In the multidimensional case the situation is slightly more complicated. If the planes $\Pi_0, \ldots, \Pi_m$ are spanned by principal axes of inertia (such rotations are called in [19] *regular*), then the rotation is stationary. But the converse is not necessarily true (see [19]). However, as it is shown in [20], the rotations which are not regular are always unstable. Therefore, it is only necessary to consider regular stationary rotations.

5.4. Mass tensor of a rigid body

From the dynamical point of view a rigid body is characterized by its mass tensor J. The entries of this tensor are given by

$$J_{ij} = \int (x_i - \widehat{x}_i)(x_j - \widehat{x}_j)\mathrm{d}\mu, \tag{13}$$

where $\widehat{x}_i$ are the center of mass coordinates.

A body is called asymmetric if all the eigenvalues of J are distinct.

5.5. Parabolic diagram of a regular stationary rotation

Consider a regular stationary rotation. Then, by definition, the planes Π_i entering (12) are spanned by principal axes of inertia. For each plane Π_i, $i > 0$ let us denote by $\lambda_1(\Pi_i), \lambda_2(\Pi_i)$ the eigenvalues of the mass tensor J corresponding to the principal axes of inertia which span Π_i. By $\omega(\Pi_i)$ denote the angular velocity of rotation in the plane Π_i.

Draw a coordinate plane. Mark squares of all eigenvalues of J on the horizontal axis. For each Π_i draw a parabola through $\lambda_1(\Pi_i)^2, \lambda_2(\Pi_i)^2$ given by $y = \chi_i(x)$, where

$$\chi_i(x) = \frac{(x - \lambda_1(\Pi_i)^2)(x - \lambda_2(\Pi_i)^2)}{\omega(\Pi_i)^2(\lambda_1(\Pi_i) + \lambda_2(\Pi_i))^2}. \tag{14}$$

For all fixed principal axes draw vertical lines through the squares of corresponding eigenvalues of J.

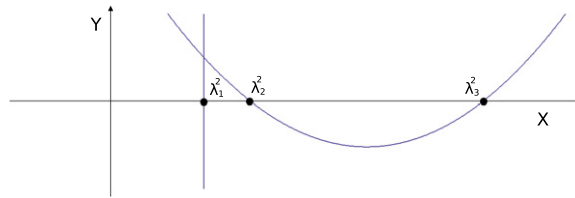[6] Note that Π_0 may be zero in the even-dimensional case, which means that there are no fixed axes.

Fig. 1. Stable rotation of a three-dimensional body around the short principal axis of inertia. $\lambda_1 < \lambda_2 < \lambda_3$ are the eigenvalues of the mass tensor.
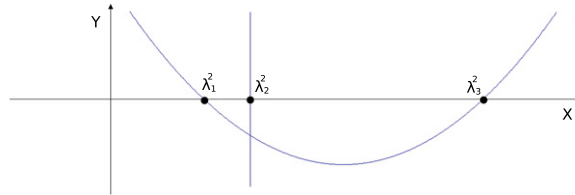

Fig. 2. Unstable rotation of a three-dimensional body around the middle principal axis of inertia.

Definition 11. The obtained picture is called *the parabolic diagram* of a regular stationary rotation.

Figs. 1 and 2 illustrate two examples of parabolic diagrams.

Definition 12. 1. Two parabolas on a parabolic diagram are said to intersect at infinity if they have only one point of intersection (of multiplicity one) or no points of intersection (neither real, nor complex).
2. Two parabolas on a parabolic diagram are said to be tangent at infinity if they have no points of intersection (real or complex).

5.6. Stability theorems

Applying Theorem 4 we obtain the following result:

Theorem 6. *Consider a regular stationary rotation of an asymmetric multidimensional rigid body. Assume that*

1. *All intersections on the parabolic diagram of the rotation are either real and belong to the upper half-plane or infinite.*
2. *There are no points of tangency on the parabolic diagram.*
3. *The rotation has no more than two fixed axes (dim $\Pi_0 \leq 2$).*

Then the rotation is stable.

Remark 8. Vice versa, if the parabolic diagram of a rotation contains at least one complex intersection point or an intersection at the lower half-plane, then the rotation is unstable. This is proved in [20].

For example, the rotation illustrated in Fig. 1 is stable, in Fig. 2—unstable.

The proof of Theorem 6 is the formal application of Theorem 4. Below are some brief comments on how the conditions of Theorem 6 are related to the conditions of Theorem 4.

1. Condition 1 of Theorem 4 which reads "rank $\Pi(x) = $ rank Π" is equivalent to the condition that "the rotation has no more than two fixed axes".
2. Condition 2 of Theorem 4 which reads "the equilibrium is regular" is equivalent to the fact that a rotation is regular.
3. The spectrum of the pencil is exactly the set of the horizontal coordinates of the intersection points on the parabolic diagram. Thus, parabolic diagrams naturally appear in the problem.
4. Condition 3 of Theorem 4 which reads "the pencil is diagonalizable" is equivalent to the condition that "there are no points of tangency on the parabolic diagram".
5. Condition 4 of Theorem 4 which reads "for each $\lambda \in \Lambda_\Pi(x)$ the λ-linearization $d_\lambda\Pi(x)$ is compact" is equivalent to the condition that "All intersections on the parabolic diagram of the rotation are either real and belong to the upper half-plane or infinite".

Note that parabolic diagrams, which appear naturally as spectral data of the Poisson pencil associated with a rigid body, give a visual interpretation of stability results even in the four-dimensional case, which was studied earlier by direct methods in [13,15,16].

Thus the biHamiltonian approach, in this case, not only allows simpler calculations but also provides a more natural interpretation of the results.

6. Proof of the stability theorem

A technique similar to that used in [7] will be used to prove Theorem 4, however the proof is self-contained.

6.1. Step 1. The forms Q_f

For notational simplicity denote the spectrum of Π at x by Λ and the cotangent space to the ambient manifold M by V^*.

Suppose that the equilibrium point x is regular. Then, by definition, $\mathrm{Ker}\,P_\alpha = \mathrm{Ker}\,P_\beta$ for all $\alpha, \beta \notin \Lambda$. Denote this common kernel by K. The regularity condition also implies that the symplectic leafs of all brackets P_α, $\alpha \notin \Lambda$ have a common tangent space at the point x. Denote this tangent space by O. It will be proved that under the conditions of Theorem 4 there exists an integral f such that $\mathrm{d}^2 f|_O(x) > 0$. If this is so, then stability follows from Theorem 2.

Without loss of generality assume that $\infty \notin \Lambda$. Then the map $P_\infty : V^*/K \to O$ is an isomorphism. Instead of $\mathrm{d}^2 f$ consider the form Q_f defined on V^*/K by

$$Q_f(\xi, \eta) = \mathrm{d}^2 f(P_\infty \xi, P_\infty \eta). \tag{15}$$

Obviously, $\mathrm{d}^2 f$ and Q_f are both simultaneously positive definite.

Let α_0 be larger than all elements of Λ. Define $\mathcal{F}$ as the space spanned by all (local) Casimir functions of all brackets P_α, where $\alpha_0 < \alpha < +\infty$. It will be proved that there exists $f \in \mathcal{F}$ such that $Q_f > 0$ on V^*/K.

6.2. Step 2. Decomposition of V^*/K

Since $\mathrm{Ker}\,P_\alpha \supset K$ for each α, all forms P_α are well defined on V^*/K. Since P_∞ is non-degenerate on V^*/K, consider the recursion operator

$$R = P_\infty^{-1} P_0 : V^*/K \to V^*/K. \tag{16}$$

It is easy to see that the spectrum of R coincides with the spectrum of the pencil: $\sigma(R) = \Lambda$. The λ-eigenspace of R is

$$V_\lambda = \mathrm{Ker}\,P_\lambda/K. \tag{17}$$

Lemma 1. *Under the conditions of Theorem 4 the operator R is diagonalizable over $\mathbb{R}$.*

Proof. First, $\Lambda \subset \mathbb{R}$ (see Remark 5). Consequently, all eigenvalues of R are real and it suffices to prove that R is diagonalizable. Suppose the contrary, i.e. that R has a Jordan block. Then there exists $\xi \in V_\lambda$, $\eta \in V^*/K$ such that $R\eta = \lambda\eta + \xi$. Therefore, for all $\zeta \in V^*/K$, $P_\lambda(\eta, \zeta) = P_\infty(\xi, \zeta)$. Consequently, $P_\infty(\xi, \zeta) = 0$ for all $\zeta \in V_\lambda$. But the diagonalizability condition implies that P_∞ is non-degenerate on V_λ. The obtained contradiction proves the lemma. $\square$

So, under the conditions of Theorem 4 there exists a decomposition

$$V^*/K = \bigoplus_{\lambda \in \Lambda} V_\lambda. \tag{18}$$

It will be shown that all forms Q_f, $f \in \mathcal{F}$ respect this decomposition.

Lemma 2. *For each $f \in \mathcal{F}$ and each $\alpha < \alpha_0$ there exist functions $\widetilde{f}, \bar{f} \in \mathcal{F}$ such that for any function g the following "recursion" relations hold:*

$$\{f, g\}_\infty = \{\widetilde{f}, g\}_\alpha, \qquad \{f, g\}_\alpha = \{\bar{f}, g\}_\infty. \tag{19}$$

Proof. First, let f be a Casimir function of P_β, $\beta > \alpha_0$. Then

$$\{f, g\}_\alpha = \{f, g\}_\beta + (\beta - \alpha)\{f, g\}_\infty = (\alpha - \beta)\{f, g\}_\infty. \tag{20}$$

Thus, $\widetilde{f} = f/(\beta - \alpha)$, $\bar{f} = (\beta - \alpha)f$ are as required.

For an arbitrary $f \in \mathcal{F}$ the statement is true by linearity. $\square$

Let $D_f P_\alpha$ be the operator dual to the linearization of $P_\alpha \mathrm{d}f$ at x. Then it is easy to see that Q_f is given by the formula

$$Q_f(\xi, \eta) = P_\infty(D_f P_\infty(\xi), \eta). \tag{21}$$

The operator $D_f P_\alpha : V^* \to V^*$ can be given by an explicit formula

$$D_f P_\alpha(\mathrm{d}g(x)) = \mathrm{d}\{f, g\}_\alpha(x). \tag{22}$$

Note that this formula, together with the Jacobi identity, implies that $D_f P_\alpha$ is skew-symmetric with respect to P_α.

Lemma 3. *For $f \in \mathcal{F}$ the operator $D_f P_\infty$ is skew-symmetric with respect to all forms P_α.*

Proof. First, $D_f P_\infty$ is skew-symmetric with respect to P_∞. Taking into account (22) and Lemma 2, $D_f P_\infty$ can be rewritten as $D_{\bar{f}} P_0$. Thus, this operator is skew-symmetric with respect to P_∞ and P_0, and, by linearity, with respect to all forms of the pencil. $\square$

Lemma 4. *For $f \in \mathcal{F}$ the recursion operator is symmetric with respect to Q_f:*

$$Q_f(R\xi, \eta) = Q_f(\xi, R\eta), \tag{23}$$

and, consequently, the summands of (18) *are pairwise orthogonal with respect to Q_f.*

Proof. Lemma 3 implies that $D_f P_\infty$ commutes with R. Also note that R is symmetric with respect to P_∞, and $D_f P_\infty$ is skew-symmetric with respect to P_∞. Therefore

$$\begin{aligned} Q_f(R\xi, \eta) &= P_\infty(D_f P_\infty(R\xi), \eta) = -P_\infty(R\xi, D_f P_\infty \eta) \\ &= -P_\infty(\xi, R D_f P_\infty(\eta)) = P_\infty(D_f P_\infty(R\eta), \xi) = Q_f(R\eta, \xi). \quad \square \end{aligned} \tag{24}$$

6.3. Step 3. Positivity of Q_f on V_λ

Lemma 5. *Under the conditions of Theorem 4 for each $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}$ such that Q_{f_λ} is positive on V_λ.*

Proof. Let $[,]$ be the commutator in $\mathfrak{g}_\lambda = \operatorname{Ker} P_\lambda$. The compactness condition implies that there exists $\nu \in K$ such that

$$A^\nu(\xi, \xi) = P_\infty([\nu, \xi], \xi) \tag{25}$$

is positive definite on V_λ. Take $f \in \mathcal{F}$ such that $df(x) = \nu$. By Lemma 2 there exists $\bar{f} \in \mathcal{F}$ such that $\{f, g\}_\lambda = \{\bar{f}, g\}_\infty$. Take $f_\lambda = \bar{f}$. Let $\xi \in V_\lambda$. Then

$$Q_{\bar{f}}(\xi, \xi) = P_\infty(D_{\bar{f}} P_\infty(\xi), \xi) = P_\infty(D_f P_\lambda(\xi), \xi). \tag{26}$$

Since $\xi \in \operatorname{Ker} P_\lambda$, (22) implies that $D_f P_\lambda(\xi) = [df(x), \xi]$. Thus,

$$Q_{\bar{f}}(\xi, \xi) = P_\infty([df, \xi], \xi) = P_\infty([\nu, \xi], \xi) = A^\nu(\xi, \xi) > 0. \quad \square \tag{27}$$

6.4. Step 4. Recursion invariance

Lemma 6. *Suppose that $f \in \mathcal{F}$ and $p(z)$ is a polynomial. Then there exists $\widetilde{f} \in \mathcal{F}$ such that*

$$Q_f(p(R)\xi, \eta) = Q_{\widetilde{f}}(\xi, \eta). \tag{28}$$

Proof. First suppose that $p(z) = z$. Since $df \in K$, there exists a Casimir function f_∞ of P_∞ such that $df(x) = df_\infty(x)$. Formula (22) implies that $D_f P_\infty = D_{f - f_\infty} P_\infty$. Further, for any g such that $dg = 0$ the same formula (22) implies that $D_g P_\alpha = d^2 g P_\alpha$, thus

$$D_f P_\infty = d^2(f - f_\infty) P_\infty. \tag{29}$$

Consequently, taking into account (21),

$$\begin{aligned} Q_f(R\xi, \eta) &= P_\infty(d^2(f - f_\infty) P_\infty R(\xi), \eta) = P_\infty(d^2(f - f_\infty) P_0(\xi), \eta) \\ &= P_\infty(D_{f - f_\infty} P_0(\xi), \eta). \end{aligned} \tag{30}$$

Note that if $f - f_\infty \in \mathcal{F}$, then, by Lemma 2, $D_{f - f_\infty} P_0(\xi)$ can be rewritten as $D_{\bar{f}} P_\infty$, which proves the lemma. However, $f - f_\infty$ is not in $\mathcal{F}$ a priori,[7] therefore the following limit argument is applied.

Choose a family f_α such that f_α is a Casimir function of P_α and $f_\alpha \to f_\infty$ as $\alpha \to \infty$. Then $f - f_\alpha \in \mathcal{F}$, and, by Lemma 2, there exists $\bar{f}_\alpha \in \mathcal{F}$ such that $\{f - f_\alpha, g\}_0 = \{\bar{f}_\alpha, g\}_\infty$. Thus, $D_{f - f_\alpha} P_0 = D_{\bar{f}_\alpha} P_\infty$. So (30) gives

$$Q_f(R\xi, \eta) = \lim_{\alpha \to \infty} P_\infty(D_{f - f_\alpha} P_0(\xi), \eta) = \lim_{\alpha \to \infty} P_\infty(D_{\bar{f}_\alpha} P_\infty(\xi), \eta) = \lim_{\alpha \to \infty} Q_{\bar{f}_\alpha}(\xi, \eta). \tag{31}$$

Consequently, the form $Q_f(R\xi, \eta)$ belongs to the closure of the space $\{Q_g, g \in \mathcal{F}\}$. But this latter space is finite-dimensional, thus $Q_f(R\xi, \eta) = Q_{\widetilde{f}}(\xi, \eta)$ for some $\widetilde{f} \in \mathcal{F}$.

For an arbitrary polynomial the lemma is proved by induction. $\square$

---

[7] This can be overcome by adding Casimir functions of P_∞ to $\mathcal{F}$. However, this would make the proof of Lemma 2 much more complicated.

6.5. Step 5. Completion of the proof

Let

$$p_{\lambda_0}(z) = \prod_{\lambda \in \Lambda \setminus \{\lambda_0\}} \frac{z - \lambda}{\lambda_0 - \lambda}. \tag{32}$$

Then $p_\lambda(R)$ is the projector $V^*/K \to V_\lambda$.

By Lemma 5 there exists $f_\lambda \in \mathcal{F}$ such that Q_{f_λ} is positive on V_λ. By Lemma 6 there exists $\widetilde{f}_\lambda \in \mathcal{F}$ such that

$$Q_{f_\lambda}(p_\lambda(R)\xi, \xi) = Q_{\widetilde{f}_\lambda}(\xi, \xi). \tag{33}$$

Take

$$f = \sum_{\lambda \in \Lambda} \widetilde{f}_\lambda. \tag{34}$$

Now claim that $Q_f > 0$ on V^*/K. By Lemma 4 it suffices to show that Q_f is positive on each V_λ, $\lambda \in \Lambda$. Let $\xi \in V_{\lambda_0}$. Then

$$Q_f(\xi, \xi) = \sum_{\lambda \in \Lambda} Q_{\widetilde{f}_\lambda}(\xi, \xi) = \sum_{\lambda \in \Lambda} Q_{f_\lambda}(p_\lambda(R)\xi, \xi) = Q_{f_{\lambda_0}}(\xi, \xi) > 0. \tag{35}$$

In the last equality we used the fact that $p_\lambda(R)$ is a projector. The theorem is proved.

References

 [1] F. Magri, A simple model of the integrable Hamiltonian equation, Journal of Mathematical Physics 19 (1978) 1156–1162.
 [2] I. Gel'fand, I. Dorfman, Hamiltonian operators and algebraic structures related to them, Functional Analysis and its Applications 13 (1979) 248–262.
 [3] A. Reiman, M. Semenov-Tyan-Shanskii, A family of Hamiltonian structures, hierarchy of Hamiltonians, and reduction for first-order matrix differential operators, Functional Analysis and its Applications 14 (1980) 146–148.
 [4] V. Arnold, B. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag, 1998.
 [5] A. Bolsinov, Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution, Mathematics of the USSR-Izvestiya 38 (1992) 69–90.
 [6] A. Bolsinov, A. Oshkov, Bi-Hamiltonian structures and singularities of integrable systems, Regular and Chaotic Dynamics 14 (2009) 431–454.
 [7] A. Bolsinov, A. Izosimov, Singularities of bi-Hamiltonian systems, 2012. arXiv:1203.3419v1.
 [8] V. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, 1978.
 [9] A. Mishchenko, A. Fomenko, Euler equations on finite-dimensional Lie groups, Mathematics of the USSR-Izvestiya 12 (1978) 371–389.
[10] I.M. Gel'fand, I.S. Zakharevich, Spectral theory of a pencil of skew-symmetric differential operators of third order on S^1, Functional Analysis and its Applications 23 (1989) 85–93.
[11] A. Bolsinov, A. Fomenko, Integrable Hamiltonian Systems. Geometry, Topology and Classification, CRC Press, 2004.
[12] A. Oshemkov, The topology of surfaces of constant energy and bifurcation diagrams for integrable cases of the dynamics of a rigid body on $\mathfrak{so}(4)$, Russian Mathematical Surveys 42 (1987) 241–242.
[13] L. Fehér, I. Marshall, Stability analysis of some integrable Euler equations for SO(n), Journal of Nonlinear Mathematical Physics 10 (2003) 304–317.
[14] A. Spiegler, Stability of generic equilibria of the 2N dimensional free rigid body using the energy-Casimir method, Ph.D. Thesis, University of Arizona, 2006.
[15] P. Birtea, I. Caşu, T. Ratiu, M. Turhan, Stability of equilibria for the $\mathfrak{so}(4)$ free rigid body, Journal of Nonlinear Science 22 (2012) 187212.
[16] P. Birtea, I. Caşu, Energy methods in the stability problem for the $\mathfrak{so}(4)$ free rigid body, 2011. arXiv:1010.0295v2.
[17] I. Caşu, On the stability problem for the $\mathfrak{so}(5)$ free rigid body, 2011. arXiv:1009.2387v2.
[18] S. Manakov, Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body, Functional Analysis and its Applications 10 (1976) 328–329.
[19] A. Izosimov, A note on relative equilibria of multidimensional rigid body, Journal of Physics A: Mathematical and Theoretical 45 (2012) 325203.
[20] A. Izosimov, Stability of relative equilibria of multidimensional rigid body, 2012. arXiv:1203.3985v1.