Stability of relative equilibria of multidimensional rigid body

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Abstract

It is a classical result of Euler that the rotation of a torque-free three-dimensional rigid body about the short or the long axis is stable, whereas the rotation about the middle axis is unstable. This result is generalized to the case of a multidimensional body.

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1. Introduction

1.1. Three-dimensional free rigid body

The Euler problem in rigid body dynamics is one of the following equivalent problems.

(1) The motion of a rigid body fixed at the centre of mass under no external forces.
(2) The motion of a rigid body which is free to move in space under no external forces.

The second problem is reduced to the first one by passing to the coordinate system related to the centre of mass. In both cases we can add a constant gravity field because the resulting torque of the gravity force with respect to the centre of mass vanishes.

   Let us consider the problem of motion of a rigid body fixed at the centre of mass acted on by no external forces. Then, as was observed by Euler, the evolution equations for the

angular velocity do not involve the position coordinates of the body. Euler's equations have the form

$$
\begin{aligned}
I_1\,\dot\omega_1 &= (I_2 - I_3)\,\omega_2\omega_3, \\
I_2\,\dot\omega_2 &= (I_3 - I_1)\,\omega_3\omega_1, \\
I_3\,\dot\omega_3 &= (I_1 - I_2)\,\omega_1\omega_2,
\end{aligned}
\tag{1}
$$

where I_1, I_2, I_3 are the principal moments of inertia and $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector written in principal axes. In terms of modern geometric mechanics, this system is obtained from the 'rigid body fixed at the centre of mass' problem by reduction with respect to the SO(3) action.

Remark 1.1. Recall that the inertia tensor of a rigid body is a positive-definite quadratic form $\mathbb{I}$ which characterizes the distribution of mass in the body. The eigenvalues of $\mathbb{I}$ are called principal moments of inertia, and its eigenvectors are called principal axes. For a uniform box-shaped body, principal axes coincide with the axes of symmetry, and principal moments are inversely proportional to the lengths of these axes. See [1] for details.

If we assume that the body is asymmetric, i.e. if I_1, I_2, and I_3 are pairwise distinct, then the right-hand sides of equation (1) vanish simultaneously if and only if the angular velocity vector is collinear to one of the three principal axes. Thus the fixed point set of the system (1) consists of three mutually orthogonal straight lines. These fixed points are stationary, or permanent, rotations of the body, i.e. such motions that the axis of rotation is time-independent. Stationary rotations are also called relative equilibria.

As was shown by Euler, stationary rotations about different principal axes have different dynamical features. Rotation about the axis of greatest moment of inertia or axis of least moment of inertia is stable, whereas rotation about the intermediate axis is unstable. This can be demonstrated by trying to spin a book about one of its symmetry axes. While the book spins fairly well about the longest and the shortest axis, spinning about the intermediate axis causes the book to 'tumble', periodically reversing the direction of rotation.

The aim of this paper is to establish a multidimensional generalization of this result. The problem was studied by a number of authors [2–7], however the general answer has only been obtained in dimension four. As the dimension grows, the problem becomes too complicated from the computational point of view when being approached by direct methods. In this paper, the problem is solved in arbitrary dimension by means of algebraic technique related to compatible Poisson brackets and Lie algebras.

1.2. Multidimensional rigid body

The possibility to generalize the free rigid body equations to the n-dimensional case was already mentioned by Frahm [8] and Weyl [9]. Arnold [1] observed that, after the standard identification of $\mathbb{R}^3$ with the space of skew-symmetric 3×3 matrices $\mathfrak{so}(3)$, equations (1) can be rewritten in the form

$$
\begin{aligned}
\dot M &= [M, \Omega], \\
M &= \Omega J + J\Omega,
\end{aligned}
\tag{2}
$$

where $M \in \mathfrak{so}(3)$ is the angular momentum, $\Omega \in \mathfrak{so}(3)$ is the angular velocity, and $J = \mathrm{diag}(J_1, J_2, J_3)$ is a constant positive-definite diagonal matrix such that $I_1 = J_2 + J_3$, $I_2 = J_1 + J_3$ and $I_3 = J_1 + J_2$.

The multidimensional generalization of equations (2) is straightforward: we just replace 3×3 matrices by $n \times n$ matrices. A somewhat better approach is to generalize not the equations but the problem. Consider an n-dimensional rigid body fixed at the centre of mass acted on by

no external forces. Fix a space frame and a body frame both centred at the centre of mass of the body. Let $X(t) \in SO(n)$ be the position of the body frame with respect to the space frame. Define $\Omega = X^{-1}\dot{X}$. This matrix is skew-symmetric and is called the angular velocity matrix. Define also a symmetric matrix J by

$$J_{ij} = \int x_i x_j \, \mathrm{d}\mu,$$

where the coordinates x_i are related to the body frame, and $\mathrm{d}\mu$ is the density of the mass distribution. Then it can be proved that the evolution of the angular velocity matrix Ω is governed by equations (2). Note that equations (2) are equivalent to the conservation of the angular momentum in the space frame

$$\frac{\mathrm{d}}{\mathrm{d}t}(XMX^{-1}) = 0.$$

See [1, 10] for details.

Remark 1.2. Following [11], we suggest that J is called the mass tensor. The mass tensor should not be confused with the inertia tensor. The inertia tensor is the map $\mathbb{I}: \mathfrak{so}(n) \to \mathfrak{so}(n)$, which is given by $\mathbb{I}(\Omega) = J\Omega + \Omega J$. In three dimensions $\mathfrak{so}(3)$ may be identified with $\mathbb{R}^3$, which may lead to a confusion between $\mathbb{I}$ and J. In higher dimensions these two operators act on different spaces.

1.3. Multidimensional rigid body as a completely integrable system

Arnold showed that the system (2) is Hamiltonian with respect to the Lie–Poisson bracket on the dual of the Lie algebra $\mathfrak{so}(n)$, and therefore the invariants of the coadjoint representation are the first integrals of the system. These first integrals are trivial in the sense that they are Casimir functions of the Lie–Poisson bracket and do not correspond to symmetries. Later, Mishchenko [12] found a family of non-trivial quadratic first integrals. They were shown to be involution with respect to the Lie–Poisson bracket by Dikii [13]. Dikii also observed that in the four-dimensional case Mishchenko's first integrals are sufficient for complete Liouville integrability. In his famous paper [14], Manakov showed that equations (2) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(M + \lambda J^2) = [M + \lambda J^2, \Omega + \lambda J],$$

which implies that the functions $f_{\lambda,k} = \mathrm{Tr}\,(M + \lambda J^2)^k$ are first integrals[1]. Later it was proved by Mishchenko and Fomenko [15] and Ratiu [10] that these first integrals Poisson-commute and are sufficient for complete Liouville integrability.

Note that we will not be using the integrals of the system in their explicit form: they are complicated polynomials not easy to deal with. Instead of considering the integrals, we will make use of the bi-Hamiltonian structure of the system, which encodes all the information about them. The bi-Hamiltonian structure of (2) was discovered by Bolsinov [16, 17][2].

1.4. Stability for the multidimensional rigid body

Below we discuss what is known about stability of stationary rotations in the multidimensional case. A more detailed comparison of previously known results with the results of this paper can be found in section 2.3.

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[1] Manakov also showed that the system (2) can be embedded into a large class of integrable systems which are now called Manakov tops. Note that all the results of this paper remain true for all generic Manakov tops.

[2] The same bi-Hamiltonian structure was later rediscovered by Morosi and Pizzocchero [18].

The first topological description of the four-dimensional rigid body problem was obtained by Oshemkov [2], who constructed bifurcation diagrams of the moment map. As it is clear now, these diagrams can be used, in principle, to study stability of stationary rotations[3].

The solution of the stability problem in dimension four was obtained by Fehér and Marshall [3], and later by another approach by Birtea and Caşu [5], Birtea *et al* [7]. In this paper these results receive a geometric interpretation (see example 2.3).

There was also an attempt to solve the stability problem in five dimensions, however only partial results are available, see Caşu [6].

The multidimensional situation was studied in the thesis of Spiegler [4]. He gave a sufficient condition for a stationary rotation to be stable in arbitrary even dimension. However, as follows from the results of this paper, this condition is far from necessary and sufficient (see example 2.5).

In this paper, the stability problem is solved almost completely in arbitrary dimension by means of the bi-Hamiltonian approach. The bi-Hamiltonian approach for studying topology and stability in integrable systems was suggested and developed in [21–23].

We also refer the reader to the author's preprint [24] where the stability problem for the multidimensional rigid body is studied by means of algebraic geometry.

1.5. Structure of the paper

The paper is organized as follows. All main results are contained in section 2. Section 2.1 is devoted to the classification of stationary rotations. In section 2.2, the notion of a parabolic diagram is defined and the main stability theorem is formulated. Section 2.3 contains some examples and compares the results of the paper to previously known results. In section 3, the machinery which allows to prove stability in bi-Hamiltonian systems is presented. In section 4, the bi-Hamiltonian structure of the multidimensional rigid body is introduced. The proof of the main theorem is in sections 5 and 6. Finally, the appendix contains the explicit classification of Lie algebras $\mathfrak{g}_\lambda$, which arise in the bi-Hamiltonian geometry of the multidimensional rigid body.

2. Main results

2.1. Stationary rotations

We study the equations

$$
\begin{aligned}
\dot{M} &= [M, \Omega], \\
M &= \Omega J + J\Omega,
\end{aligned}
\tag{3}
$$

where $M \in \mathfrak{so}(n)$ is called the angular momentum matrix[4], $\Omega \in \mathfrak{so}(n)$ is called the angular velocity matrix and J is a positive symmetric matrix called the mass tensor. Without loss of generality we may assume that J is diagonal.

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[3] See [19] where the relation between topology and stability in integrable systems is discussed. Also note that the topological approach to integrable systems was first proposed in the classical work of Smale [20].

[4] To be precise, the angular momentum M belongs to the dual space $\mathfrak{so}(n)^*$. In what follows, we identify $\mathfrak{so}(n)$ and $\mathfrak{so}(n)^*$ by means of the Killing form $\operatorname{Tr} XY$.

Before describing stationary rotations, consider how an n-dimensional body may rotate. At each moment of time, the angular velocity matrix Ω may be brought to a canonical form

$$
\Omega = \begin{pmatrix}
0 & \omega_1 & & & & & \\
-\omega_1 & 0 & & & & & \\
& & \ddots & & & & \\
& & & 0 & \omega_m & & \\
& & & -\omega_m & 0 & & \\
& & & & & 0 & \\
& & & & & & \ddots
\end{pmatrix}
\tag{4}
$$

by an orthogonal transformation. In other words, $\mathbb{R}^n$ is decomposed into a sum of m pairwise orthogonal two-dimensional planes $\Pi_1, \dots, \Pi_m$ and a space Π_0 of dimension $n - 2m$ orthogonal to all these planes:

$$
\mathbb{R}^n = \left(\bigoplus_{i=1}^{m} \Pi_i \right) \oplus \Pi_0.
\tag{5}
$$

There is an independent rotation in each of the planes $\Pi_1, \dots, \Pi_m$, while $\Pi_0 = \mathrm{Ker}\,\Omega$ is immovable.

Definition 1. The eigenvectors of J are called *principal axes of inertia*.

A classical three-dimensional result states that a rotation is stationary if and only if it is a rotation about a principal axis of inertia. In the multidimensional case, this is not always so. Stationary rotations of a multidimensional rigid body are described in [25].

Proposition 2.1. *Consider the system ([3]). Suppose that J has pairwise distinct eigenvalues. Then M is an equilibrium point of the system if and only if there exists an orthonormal basis such that J is diagonal, and Ω is block-diagonal of the following form*

$$
\Omega = \begin{pmatrix}
\omega_1 \Omega_1 & & & & \\
& \ddots & & & \\
& & \omega_k \Omega_k & & \\
& & & 0 & \\
& & & & \ddots
\end{pmatrix},
$$

where $\Omega_i \in \mathfrak{so}(2m_i) \cap \mathrm{SO}(2m_i)$ for some $m_i > 0$, and $\omega_i s$ are distinct positive real numbers.

Definition 2. A stationary rotation M is *regular* if there exists an orthonormal basis such that J is diagonal, and Ω is of the form ([4]). Otherwise, M is *exotic*.

In other words, a rotation is regular if all the planes Π_i entering ([5]) are spanned by principal axes of inertia. Note that if all non-zero eigenvalues of Ω are distinct, then the rotation is automatically regular. In the three-dimensional case, all stationary rotations are regular.

In this paper, only regular stationary rotations are considered.

Remark 2.1. It was asserted in the announcement [26] as well as in the earlier version of this paper that all exotic stationary rotations are unstable, however some technical details in the
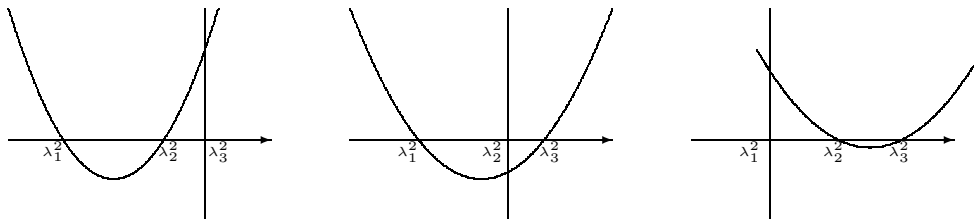
Figure 1. Parabolic diagrams for the three-dimensional rigid body. Rotations around the long, middle and short axes of inertia, respectively.

proof are still to be completed[5]. The proof will be published elsewhere. Also note that exotic stationary rotations arise as relative equilibria of the *n* point masses problem [28].

The main problem of the paper is to study regular stationary rotations for Lyapunov stability. It will be assumed that the body is *asymmetric*, i.e. all eigenvalues of J are pairwise distinct.

2.2. Parabolic diagrams and stability

Consider a regular stationary rotation. Then there exists an orthonormal basis such that J is diagonal, and Ω is given by (4). In other words, there exists a decomposition (5) in which all planes Π_i are spanned by principal axes of inertia.

  Define the notion of the parabolic diagram of a regular stationary rotation.

(1) Draw a coordinate plane.
(2) For each 2-plane Π_i, $i > 0$, draw the parabola given by $y = \chi_i(x)$ where

$$\chi_i(x) = \frac{(x - \lambda_{i1}^2)(x - \lambda_{i2}^2)}{\omega_i^2(\lambda_{i1} + \lambda_{i2})^2}, \tag{6}$$

  ω_i is the frequency of rotation in the plane Π_i and $\lambda_{i1}, \lambda_{i2}$ are the eigenvalues of J corresponding to the eigenvectors $e_{i1}, e_{i2} \in \Pi_i$.
(3) For each immovable principal axis $e_0 \in \Pi_0$, draw a vertical straight line through λ^2 where λ is the eigenvalue of J corresponding to the eigenvector e_0.

As a result, there is either a parabola or a vertical straight line passing through the square of each eigenvalue of J.

Definition 3. The obtained picture is the *parabolic diagram* of a regular stationary rotation.

  Parabolic diagrams for the three-dimensional rigid body are depicted in figure 1. See section 2.3 for more examples.

  The following theorem is the main result of the paper.

Theorem 1. *Consider a regular stationary rotation of a multidimensional rigid body.*

(1) Assume that

  (a) all intersections in the associated parabolic diagram are either real and belong to the upper half-plane, or infinite;

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[5] Namely, we can prove that all exotic equilibria are unstable provided that (3) is known to be a non-resonant system, which means that trajectories of the system are dense on almost all Liouville tori. Moreover, we can prove that the non-resonant condition is satisfied for a certain open subset of the phase space. Since the system is analytic, this should imply that it is non-resonant everywhere [27], however the proof of this latter assertion is unknown to the author.
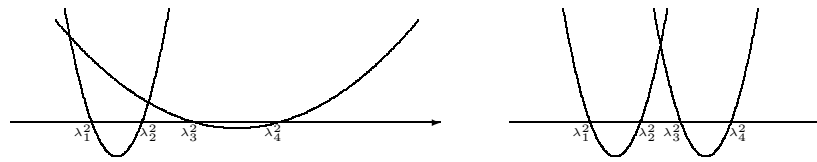
Figure 2. Parabolic diagrams for the four-dimensional rigid body. Stable rotation.
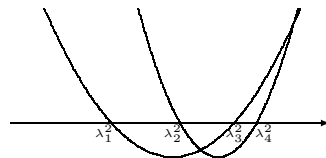

Figure 3. Parabolic diagram for the four-dimensional rigid body. Unstable rotation.

　　(b) the are no tangency points in the parabolic diagram.

　　Then the rotation is stable.

(2) *Assume that there is at least one intersection in the parabolic diagram which is either complex or belongs to the lower half-plane. Then the rotation is unstable.*

Remark 2.2. When speaking about real, complex or infinite intersections, the parabolic diagram is considered as a curve in $\mathbb{CP}^2$.

Remark 2.3. In [22], a weaker version of the first statement of theorem 1 was announced. It included an additional requirement that $\dim \Pi_0 \leqslant 2$. In this paper, the technique of [22] is extended, so that the mentioned requirement could be omitted: it seems to be quite natural to consider rotations with a large number of fixed axes.

Remark 2.4. Note that this theorem solves the stability problem for an open dense subset of regular stationary rotations.

Remark 2.5. It is proved in the preprint [24] that condition 1(b) of theorem 1 can be omitted, so a regular stationary rotation is stable if and only if all intersections in the associated parabolic diagram are either real and belong to the upper half-plane, or infinite. The proof uses methods from algebraic geometry.

2.3. Examples

Example 2.1 (Three-dimensional rigid body). Parabolic diagrams for the three-dimensional rigid body are depicted in figure 1. The classical results on stability are immediately recovered.

Example 2.2 (Four-dimensional rigid body). Let e_1, e_2, e_3, e_4 be the principal axes sorted in order of increasing eigenvalues of J. There are three possibilities.

(1) $\Pi_1 = \langle e_1, e_2 \rangle$, $\Pi_2 = \langle e_3, e_4 \rangle$. The rotation is stable (see figure 2; note that there is an intersection at infinity in the second diagram).
(2) $\Pi_1 = \langle e_1, e_3 \rangle$, $\Pi_2 = \langle e_2, e_4 \rangle$. The rotation is unstable (see figure 3).
(3) $\Pi_1 = \langle e_1, e_4 \rangle$, $\Pi_2 = \langle e_2, e_3 \rangle$. In this case, stability depends on the ratio of angular velocities. If $\omega_1 \gg \omega_2$, then the rotation is unstable (see figure 4; note that there is a complex intersection in the second diagram). If $\omega_1 \ll \omega_2$, then the rotation is stable (see figure 5).
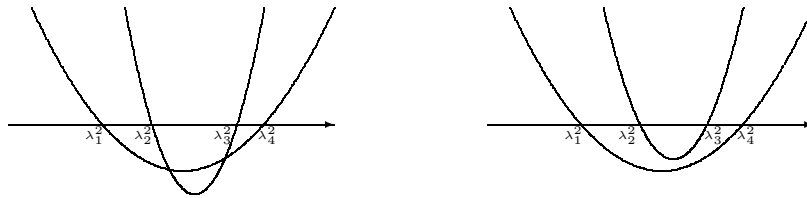
Figure 4. Parabolic diagrams for the four-dimensional rigid body. Unstable rotation.
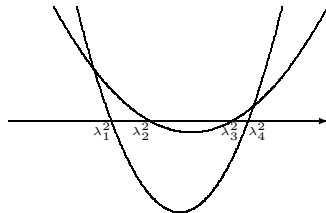

Figure 5. Parabolic diagram for the four-dimensional rigid body. Stable rotation.
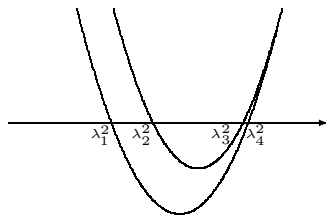
Figure 6. Parabolic diagram for the four-dimensional rigid body. Hamiltonian Hopf bifurcation.

The conclusions of items (1), (2), (3) above coincide with the results of [3,5,7]. Of course, the papers [3,5,7] do not use the language of parabolic diagrams and give stability conditions in terms of some inequalities. However, the translation from the language of parabolic diagrams to the language of inequalities is straightforward.

Note that there is a case with a tangency point in the upper half-plane (figure 6) when theorem 1 is not applicable. It is claimed in [3,7] that this rotation is unstable, however this conclusion seems to be incorrect. This follows from the results of [24] and can also be deduced from the bifurcation diagrams constructed by Oshemkov [2].

Example 2.3 (Two-dimensional rotation). Suppose that there is only one plane of rotation Π_1, i.e. the body is rotating about a subspace of codimension two. Sort principal axes in order of increasing eigenvalues of J. Then theorem 1 implies that the rotation is stable if and only if the plane of rotation is spanned by two adjacent axes: $\Pi_1 = \langle e_i, e_{i+1} \rangle$. This result can be viewed as a natural generalization of the Euler theorem.

Example 2.4 (Rotation with one fixed axis). Suppose that there is only one fixed axis, i.e. Π_0 is one-dimensional. Sort principal axes in order of increasing eigenvalues of J. Assume that the fixed axis is in the even place (2nd, or 4th, or 6th, ...). Then the rotation is unstable. This result can also be viewed as a natural generalization of Euler's theorem about instability of rotation about the middle axis.

Example 2.5 (Spiegler's theorem). Below is the main result of the work [4], reformulated in terms of this paper.

Theorem 2 (Spiegler [4]). *Consider a regular stationary rotation of a $2m$-dimensional rigid body. Sort principal axes in order of increasing eigenvalues of J. Assume that*

(1) all planes of rotation are spanned by two adjacent axes: $\Pi_1 = \langle e_1, e_2 \rangle$, $\Pi_2 = \langle e_3, e_4 \rangle$, *...;*
(2) $|\omega_1| > \ldots > |\omega_n|$, *or* $|\omega_1| < \ldots < |\omega_n|$.

Then the rotation is stable.

It is easy to see that theorem 1 implies theorem 2. Moreover, theorem 1 implies that condition 2 of theorem 2 can be omitted, and there are many more stability cases not covered by the result of Spiegler (see e.g. figure 2).

Note that Spiegler's approach to the problem is based on the method known as the Arnold energy-Casimir method (see [1, 29]). As he proves, condition of theorem 2 is necessary and sufficient for the Hessian of the energy to be positive-definite on the coadjoint orbit. By comparing theorem 2 with theorem 1, we conclude that for the majority of stable stationary rotations the Hessian of the energy is indefinite, so the energy-Casimir method fails. For these rotations, another Lyapunov function is needed to prove stability. Such a function can be explicitly found in small dimensions, as was done in [3, 5, 6], however it is not clear how to proceed for general n. The method of this paper allows to prove the existence of a Lyapunov function without finding it explicitly.

3. Bi-Hamiltonian structures and stability

In this section, basic definitions and theorems related to stability in bi-Hamiltonian systems are formulated. Most of them can be found in [22, 23]. Basic notions from Poisson geometry used throughout the section can be found in [30].

3.1. Basic notions

Definition 4. Two Poisson brackets on a manifold M are called *compatible*, if any linear combination of them is a Poisson bracket again. The *Poisson pencil* generated by two compatible Poisson brackets P_0, P_∞ is the set

$$\Pi := \{P_\lambda = P_0 - \lambda P_\infty\}_{\lambda \in \overline{\mathbb{C}}}.$$

A vector field X is bi-Hamiltonian with respect to a pencil Π if it is Hamiltonian with respect to all brackets of the pencil, i.e. for any $\lambda \in \overline{\mathbb{C}}$ there exists a (complex-valued) smooth function H_λ such that

$$X = P_\lambda \, dH_\lambda.$$

The notion of a bi-Hamiltonian system was introduced by Magri [31], Gel'fand and Dorfman [32].

Remark 3.1. For complex values of λ, the bracket P_λ should be treated as a complex-valued Poisson bracket on complex-valued functions. The corresponding Poisson tensor is a bilinear form on the complexified cotangent space at each point.

Definition 5. The *rank of a pencil* Π *at a point x* is the number

$$\operatorname{rank} \Pi(x) := \max_{\lambda \in \overline{\mathbb{C}}} \operatorname{rank} P_\lambda(x).$$

The *rank of a pencil* Π (on a manifold M) is the number

$$\operatorname{rank} \Pi := \max_{x \in M} \operatorname{rank} \Pi(x).$$

Definition 6. The *spectrum* of a pencil Π at a point x is the set

$$\Lambda_\Pi(x) := \{\lambda \in \overline{\mathbb{C}} \mid \operatorname{rank} P_\lambda(x) < \operatorname{rank} \Pi(x)\}.$$

When Π is fixed, the notation $\Lambda(x)$ is also used.

By $\mathfrak{g}_\lambda(x)$, denote the Lie algebra structure defined on $\operatorname{Ker} P_\lambda(x)$ by the linear part of P_λ at the point x. The commutator in $\mathfrak{g}_\lambda$ is given by

$$[\xi, \eta]_\lambda := \mathrm{d}\{f, g\}_\lambda(x),$$

where $\xi, \eta \in \operatorname{Ker} P_\lambda$, and f, g are any smooth functions such that $\mathrm{d}f(x) = \xi$, $\mathrm{d}g(x) = \eta$.

The algebra $\mathfrak{g}_\lambda(x)$ is mainly considered only for $\lambda \in \Lambda(x)$.

Remark 3.2. For $\lambda \in \overline{\mathbb{R}}$, the algebra $\mathfrak{g}_\lambda$ is real. However, for complex values of λ, the space $\operatorname{Ker} P_\lambda(x)$ is a subspace of $\mathrm{T}_x^* M \otimes \mathbb{C}$, and therefore $\mathfrak{g}_\lambda$ is considered as a complex Lie algebra.

Let X be a system which is bi-Hamiltonian with respect to Π, and let x be an equilibrium point of X.

Definition 7. Say that x is *regular* if the following condition holds:

$$\operatorname{Ker} P_\alpha(x) = \operatorname{Ker} P_\beta(x) \qquad \text{for all } \alpha, \beta \notin \Lambda(x).$$

Remark 3.3. Under some additional technical assumptions, regularity is equivalent to the following: x is an equilibrium for all systems which are bi-Hamiltonian with respect to the pencil Π (see [21, 22]). Equilibria not satisfying this condition are normally unstable.

For a regular equilibrium x, denote $\mathfrak{Ker}(x) := \operatorname{Ker} P_\alpha(x) = \operatorname{Ker} P_\beta(x)$. It is easy to see that $\mathfrak{Ker}(x) \subset \mathfrak{g}_\lambda(x)$ is a Lie subalgebra for each λ.

3.2. Spectral formula for bi-Hamiltonian systems

Let x be a regular equilibrium of a bi-Hamiltonian system $X = P_\lambda \, \mathrm{d}H_\lambda(x)$. Let $\xi \in \mathfrak{g}_\lambda(x) = \operatorname{Ker} P_\lambda(x)$. Denote

$$\operatorname{ad}_\lambda \xi(\eta) := [\xi, \eta]_\lambda,$$

where $[\,,\,]_\lambda$ is the commutator in $\mathfrak{g}_\lambda(x)$.

Note that $P_\lambda \, \mathrm{d}H_\lambda(x) = X(x) = 0$, so $\mathrm{d}H_\lambda(x) \in \operatorname{Ker} P_\lambda(x) = \mathfrak{g}_\lambda(x)$. Consequently, the operator $\operatorname{ad}_\lambda \mathrm{d}H_\lambda(x)$ is well-defined.

Proposition 3.1. *The subalgebra $\mathfrak{Ker}(x) \subset \mathfrak{g}_\lambda(x)$ is invariant with respect to $\operatorname{ad}_\lambda \mathrm{d}H_\lambda(x)$, i.e.*

$$[\mathrm{d}H_\lambda(x), \mathfrak{Ker}(x)]_\lambda \subset \mathfrak{Ker}(x).$$

Proof. Let $\xi \in \mathfrak{Ker}(x)$. By definition,

$$[\mathrm{d}H_\lambda(x), \xi]_\lambda = \mathrm{d}\{H_\lambda, g\}_\lambda(x),$$

where g is any function such that $\mathrm{d}g(x) = \xi$. Further,

$$\{H_\lambda, g\}_\lambda = X(g) = \{H_\alpha, g\}_\alpha$$

for any $\alpha \in \overline{\mathbb{C}}$. So,

$$[\mathrm{d}H_\lambda(x), \xi]_\lambda = \mathrm{d}\{H_\alpha, g\}_\alpha(x) = [\mathrm{d}H_\alpha, \xi]_\alpha.$$

If $\alpha \notin \Lambda(x)$, then $[\mathrm{d}H_\alpha, \xi]_\alpha \in \operatorname{Ker} P_\alpha(x) = \mathfrak{Ker}(x)$, so $[\mathrm{d}H_\lambda(x), \xi]_\lambda \in \mathfrak{Ker}(x)$. $\qquad \square$

Let x be a regular equilibrium point of a bi-Hamiltonian system. Then all symplectic leaves of generic brackets $P_\alpha, \alpha \notin \Lambda(x)$ are tangent to each other. Denote their common tangent space by $\mathrm{T}(x)$.

The following statement is used to find the spectrum of a bi-Hamiltonian system linearized at a regular equilibrium point.

Lemma 3.1. *Suppose that Π is a Poisson pencil on a finite-dimensional manifold, and $X = P_\lambda \, \mathrm{d}H_\lambda$ is a system which is bi-Hamiltonian with respect to Π. Let x be a regular equilibrium of X.*

Then the spectrum of the linearization of X at x restricted to $\mathrm{T}(x)$ is given by

$$\sigma(\mathrm{d}X\mid_{\mathrm{T}(x)}) = \bigcup_{\lambda \in \Lambda(x)} \sigma\left((\mathrm{ad}_\lambda \, \mathrm{d}H_\lambda(x))\mid_{\mathfrak{g}_\lambda(x)/\mathfrak{Ker}(x)}\right),$$

where $\sigma(P)$ stands for the spectrum of the operator P.

Note that the restriction of $\mathrm{ad}_\lambda \, \mathrm{d}H_\lambda(x)$ to $\mathfrak{g}_\lambda(x)/\mathfrak{Ker}(x)$ is well-defined since $\mathfrak{Ker}(x)$ is invariant with respect to $\mathrm{ad}_\lambda \, \mathrm{d}H_\lambda(x)$ (see proposition 3.1).

The proof of lemma 3.1 easily follows from the results of [22, 23].

3.3. Linearization of a Poisson pencil and nonlinear stability

Definition 8. Let $\mathfrak{g}$ be a (real or complex) Lie algebra, and let $\mathfrak{B}$ be a skew-symmetric bilinear form on $\mathfrak{g}$. Then $\mathfrak{B}$ can be considered as a Poisson tensor on the dual space $\mathfrak{g}^*$. Assume that the corresponding bracket is compatible with the Lie–Poisson bracket. In this case the Poisson pencil $\Pi(\mathfrak{g}, \mathfrak{B})$ generated by these two brackets is called the *linear pencil* associated with the pair $(\mathfrak{g}, \mathfrak{B})$.

Proposition 3.2. *A form $\mathfrak{B}$ on $\mathfrak{g}$ is compatible with the Lie–Poisson bracket if and only if this form is a Lie algebra 2-cocycle, i.e.*

$$\mathrm{d}\mathfrak{B}(\xi, \eta, \zeta) := \mathfrak{B}([\xi, \eta], \zeta) + \mathfrak{B}([\eta, \zeta], \xi) + \mathfrak{B}([\zeta, \xi], \eta) = 0$$

for any $\xi, \eta, \zeta \in \mathfrak{g}$.

Below is the central construction of the theory discussed in the present section. Let Π be an arbitrary Poisson pencil on a manifold M, and $x \in$ M. As before, denote the Lie algebra on $\mathrm{Ker}\, P_\lambda(x)$ by $\mathfrak{g}_\lambda(x)$. It turns out that apart from the Lie algebra structure, $\mathfrak{g}_\lambda$ carries one more additional structure.

Proposition 3.3.

(1) For any α and β the restrictions of $P_\alpha(x)$, $P_\beta(x)$ on $\mathfrak{g}_\lambda(x)$ coincide up to a constant factor.
(2) The 2-form $P_\alpha|_{\mathfrak{g}_\lambda}$ is a 2-cocycle on $\mathfrak{g}_\lambda$.

Consequently, $P_\alpha|_{\mathfrak{g}_\lambda}$ defines a linear Poisson pencil on $\mathfrak{g}_\lambda^*$. Since $P_\alpha|_{\mathfrak{g}_\lambda}$ is defined up to a constant factor, the pencil is well-defined. Denote this pencil by $\mathrm{d}_\lambda \Pi(x)$.

Definition 9. The pencil $\mathrm{d}_\lambda \Pi(x)$ is called the λ-*linearization* of the pencil Π at x.

Now, let $\mathfrak{B}$ be a 2-cocycle on a Lie algebra $\mathfrak{g}$. For an arbitrary element $\nu \in \mathrm{Ker}\,\mathfrak{B}$, define the bilinear form

$$\mathfrak{B}_\nu(\xi, \eta) := \mathfrak{B}([\nu, \xi], \eta).$$

The cocycle identity implies that this form is symmetric. Furthermore, $\mathrm{Ker}\,\mathfrak{B}_\nu \supset \mathrm{Ker}\,\mathfrak{B}$, therefore $\mathfrak{B}_\nu$ is a well-defined symmetric form on the vector space $\mathfrak{g}/\mathrm{Ker}\,\mathfrak{B}$.

Definition 10. A linear pencil $\Pi(\mathfrak{g}, \mathfrak{B})$ is *compact* if there exists $\nu \in \mathcal{Z}(\mathrm{Ker}\,\mathfrak{B})$ such that $\mathfrak{B}_\nu$ is positive-definite on $\mathfrak{g}/\mathrm{Ker}\,\mathfrak{B}$.

Remark 3.4. The notation $\mathcal{Z}(\mathfrak{g})$ stands for the centre of the Lie algebra $\mathfrak{g}$.

Definition 10 is motivated by the following statement.

Proposition 3.4. *Any linear pencil on a compact semisimple Lie algebra is compact.*

Proof. Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Since $\mathrm{H}^2(\mathfrak{g}) = 0$, any cocycle $\mathfrak{B}$ on $\mathfrak{g}$ has the form $\mathfrak{B}(\xi, \eta) = \langle \zeta, [\xi, \eta] \rangle$, where $\langle\,,\,\rangle$ is the Killing form, $\zeta \in \mathfrak{g}$. Take $\nu = \zeta$. Note that $\mathrm{Ker}\,\mathfrak{B}$ is the centralizer of ν, so $\nu \in \mathcal{Z}(\mathrm{Ker}\,\mathfrak{B})$. Further,

$$\mathfrak{B}_\nu(\xi, \xi) = \langle \nu, [[\nu, \xi], \xi] \rangle = -\langle [\nu, \xi], [\nu, \xi] \rangle > 0,$$

so the pencil is compact. $\qquad\square$

Another motivation for definition 10 is the following fact: let a system X be bi-Hamiltonian with respect to a compact linear pencil; then all trajectories of X are bounded.

In this paper, there will be non-trivial examples of compact linear pencils on non-compact Lie algebras $\mathfrak{u}(p, q)$ and $\mathfrak{u}(p, q) \ltimes \mathbb{C}^{p+q}$ arising as λ-linearizations of the pencil related to the multidimensional rigid body (see the appendix).

Definition 11. A pencil Π is called diagonalizable at x if

$$\dim \mathrm{Ker}\left(P_\alpha(x)|_{P_\lambda(x)}\right) = \mathrm{corank}\,\Pi(x) \text{ for all } \lambda \in \Lambda_\Pi(x), \alpha \neq \lambda. \tag{7}$$

Note that if (7) is satisfied for some $\alpha \neq \lambda$, then it is satisfied for any $\alpha \neq \lambda$ (see proposition 3.3).

The following theorem is used to prove nonlinear stability for a bi-Hamiltonian system.

Theorem 3. *Suppose that Π is a Poisson pencil on a finite-dimensional manifold, and X is bi-Hamiltonian with respect to Π. Let x be an equilibrium point of X. Assume that*

(1) $\mathrm{rank}\,\Pi(x) = \mathrm{rank}\,\Pi$.
(2) The equilibrium x is regular.
(3) The spectrum of Π at x is real: $\Lambda_\Pi(x) \subset \overline{\mathbb{R}}$.
(4) The pencil Π is diagonalizable at x.
(5) For each $\lambda \in \Lambda_\Pi(x)$ the λ-linearization $\mathrm{d}_\lambda \Pi(x)$ is compact.

Then x is Lyapunov stable.

See [22] for the proof.

Remark 3.5. The idea of the proof can be explained as follows. A bi-Hamiltonian system automatically possesses a large number of first integrals: these are the Casimir functions of all brackets of the pencil. The Hessians of these functions are controlled by linear parts of the corresponding brackets. The conditions of the theorem allow to show that there exists a linear combination of Casimir functions with a positive-definite Hessian, so that this combination can be used as a Lyapunov function.

3.4. Generalized stability theorem

In this section, a stronger stability result is formulated which allows to proceed for those points where rank $\Pi(x)$ < rank Π. The condition rank $\Pi(x)$ = rank Π can only be omitted under some additional technical assumptions.

Let X be a system which is bi-Hamiltonian with respect to a pencil Π, and let x be a regular equilibrium point of X. Then for each $\alpha \notin \Lambda(x)$, the linear part of $P_\alpha(x)$ defines a natural Lie algebra structure on $\mathfrak{Ker}(x) = \mathrm{Ker}\, P_\alpha(x)$. For $\lambda \in \Lambda(x)$, there is a strict inclusion $\mathfrak{Ker}(x) \subset \mathrm{Ker}\, P_\lambda(x)$. However, since $P_\lambda(x)$ is a linear combination of $P_\alpha(x)$ and $P_\beta(x)$ for any $\alpha \neq \beta \in \overline{\mathbb{C}}$, the subspace $\mathfrak{Ker}(x)$ is a subalgebra in $\mathrm{Ker}\, P_\lambda(x)$ for any λ. Thus, $\mathfrak{Ker}(x)$ carries a structure of a Lie pencil. Denote by $\mathcal{Z}_\alpha(\mathfrak{Ker}(x))$ the centre of $\mathfrak{Ker}(x)$ with respect to the Lie structure $[\ ,\ ]_\alpha$.

Definition 12. Say that x is *strongly regular* if it is regular, and

$$\mathcal{Z}(\mathfrak{Ker}(x)) := \mathcal{Z}_\alpha(\mathfrak{Ker}(x))$$

does not depend on α.

Remark 3.6. The centre of the kernel is important for the following reason: if f is a Casimir function of P, then $\mathrm{d}f(x) \in \mathcal{Z}(\mathrm{Ker}\, P(x))$. Moreover, if the transverse Poisson structure to P at the point x is linearizable, then the differentials of Casimir functions span $\mathcal{Z}(\mathrm{Ker}\, P(x))$.

Definition 13. A pencil Π is called *fine* at a point x if there exists $\alpha \notin \Lambda(x)$ and an open neighbourhood $U \ni \alpha$ such that

(1) for each $\beta \in U$ the transverse Poisson structure to P_β at the point x is linearizable, and its linear part is compact;
(2) for any $f_\alpha \in \mathcal{Z}(P_\alpha)$ there exists a family f_β depending continuously on β and defined for $\beta \in U$ such that $f_\beta \in \mathcal{Z}(P_\beta)$, i.e. any Casimir function of P_α can be 'approximated' by Casimir functions of nearby brackets of the pencil.

Remark 3.7. The notation $\mathcal{Z}(P)$ stands for the set of (local) Casimir functions of the Poisson bracket P.

The following theorem is a generalization of theorem 3.

Theorem 4. *Suppose that Π is a Poisson pencil on a finite-dimensional manifold, and X is a dynamical system which is bi-Hamiltonian with respect to Π. Let x be an equilibrium point of X. Assume that*

(1) The pencil Π is fine at x.
(2) The equilibrium point x is strongly regular.
(3) The spectrum of Π at x is real: $\Lambda_\Pi(x) \subset \overline{\mathbb{R}}$.
(4) The pencil Π is diagonalizable at x.
(5) For each $\lambda \in \Lambda_\Pi(x)$ the λ-linearization $\mathrm{d}_\lambda \Pi(x)$ is compact.

Then x is Lyapunov stable.

It is easy to see that if rank $\Pi(x)$ = rank Π, then the pencil Π is fine at x, and regularity is equivalent to strong regularity. So, theorem 4 is a generalization of theorem 3. The proof of theorem 4 repeats the proof of theorem 3.

4. Bi-Hamiltonian structure of the multidimensional rigid body

Denote the standard Lie bracket on $\mathfrak{so}(n)$ by $[\,,\,]_\infty$ and the corresponding Lie–Poisson bracket on $\mathfrak{so}(n)^*$ by $\{\,,\,\}_\infty$. The latter is given by

$$\{f, g\}_\infty(M) := \langle M, [\mathrm{d}f, \mathrm{d}g]_\infty\rangle$$

for $M \in \mathfrak{so}(n)^* \simeq \mathfrak{so}(n)$ and $f, g \in \mathrm{C}^\infty(\mathfrak{so}(n)^*)$. By $\langle\,,\,\rangle$ we denote the Killing form

$$\langle X, Y\rangle = \operatorname{Tr} XY,$$

and $\mathfrak{so}(n)^*$ is identified with $\mathfrak{so}(n)$ by means of this form.

The following was observed by Arnold [1].

Proposition 4.1. *Equations ([3](#)) are Hamiltonian with respect to the bracket* $\{\,,\,\}_\infty$. *The Hamiltonian is given by the kinetic energy*

$$H_\infty := \frac{1}{2}\langle\Omega, M\rangle.$$

Now introduce a second operation on $\mathfrak{so}(n)$ defined by

$$[X, Y]_0 := X J^2 Y - Y J^2 X.$$

Proposition 4.2.

(1) $[\,,\,]_0$ *is a Lie bracket compatible with the standard Lie bracket. In other words, any linear combination of these brackets defines a Lie algebra structure on* $\mathfrak{so}(n)$.

(2) The corresponding Lie–Poisson bracket $\{\,,\,\}_0$ *on* $\mathfrak{so}(n)^*$ *given by*

$$\{f, g\}_0 := \langle M, [\mathrm{d}f, \mathrm{d}g]_0\rangle$$

is compatible with the Lie–Poisson bracket $\{\,,\,\}_\infty$.

Consequently, a Lie pencil is defined on $\mathfrak{so}(n)$, and a Poisson pencil is defined on $\mathfrak{so}(n)^*$. Write down these pencils in the form

$$[X, Y]_\lambda = [X, Y]_0 - \lambda[X, Y]_\infty = X(J^2 - \lambda E)Y - Y(J^2 - \lambda E)X \qquad (8)$$

for $X, Y \in \mathfrak{so}(n)$ and

$$\{f, g\}_\lambda = \{f, g\}_0 - \lambda\{f, g\}_\infty = \langle M, \mathrm{d}f(J^2 - \lambda E)\mathrm{d}g - \mathrm{d}g(J^2 - \lambda E)\mathrm{d}f\rangle \quad (9)$$

for $M \in \mathfrak{so}(n)^*$ and $f, g \in \mathrm{C}^\infty(\mathfrak{so}(n)^*)$.

The Poisson tensor corresponding to the bracket $\{\,,\,\}_\lambda$ reads

$$P_\lambda(M)(X, Y) = \langle M, X(J^2 - \lambda E)Y - Y(J^2 - \lambda E)X\rangle, \qquad (10)$$

where $M \in \mathfrak{so}(n)^*$ and $X, Y \in \mathrm{T}_M^*(\mathfrak{so}(n)^*) = \mathfrak{so}(n)$.

Proposition 4.3 (Bolsinov [17, 33]). *The system ([3](#)) is Hamiltonian with respect to any bracket* $\{\,,\,\}_\lambda$, *so it is bi-Hamiltonian. The Hamiltonian is given by*

$$H_\lambda := -\frac{1}{2}\langle(J + \sqrt{\lambda}E)^{-1}\Omega(J + \sqrt{\lambda}E)^{-1}, M\rangle. \qquad (11)$$

Remark 4.1. The matrix $J + \sqrt{\lambda}E$ is invertible for any λ if the proper value of the square root is chosen.

Note that the function H_λ written here is different from the one given by Bolsinov. The difference is a Casimir function of P_λ.

5. Proof of theorem 1: instability

The proof consists of the following steps:

(1) check that a regular stationary rotation (in the sense of definition 2) is a regular equilibrium (in the sense of definition 7), so that lemma 3.1 can be applied (section 5.1);
(2) describe the spectrum $\Lambda(M)$ (section 5.2);
(3) describe the adjoint operators ad_λ (section 5.3);
(4) find the spectrum of the linearized system using lemma 3.1 (section 5.4).

Fix some notation which is used throughout the proofs.

It is only regular stationary rotations which are considered. So, assume that there exists an orthonormal basis such that J is diagonal, while Ω and M are block-diagonal with two-by-two blocks on the diagonal (definition 2). Denote by λ_i the diagonal elements of J in this basis. Note that this means that λ_i are possibly different for different rotations. However, they are unique up to a permutation and coincide with the eigenvalues of J.

By ω_is, denote the non-zero entries of the matrix Ω as in (4). By $m_i = (\lambda_{2i-1} + \lambda_{2i})\omega_i$, denote the non-zero entries of the matrix M. The notation M_i stands for the diagonal two-by-two blocks of M, i.e.

$$M_i := \begin{pmatrix} 0 & m_i \\ -m_i & 0 \end{pmatrix}. \tag{12}$$

The number n stands for the dimension of the body and m stands for the number of non-zero ω_is, that is for the number of two-dimensional planes in the decomposition (5).

For a fixed λ, let $A := J^2 - \lambda E$ if $\lambda \neq \infty$ or $A := E$ otherwise. By a_i, denote the diagonal entries of the matrix A. Clearly, $a_i = \lambda_i^2 - \lambda$ if $\lambda \neq \infty$, and $a_i = 1$ otherwise. It is also convenient to represent A as

$$A = \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_m & & & & \\ & & & a_{2m+1} & & & \\ & & & & \ddots & & \\ & & & & & a_n \end{pmatrix}, \tag{13}$$

where A_i are two-by-two diagonal matrices and a_i are numbers.

Further, extend the definition of $\chi_i(x)$ given by (6) to the point ∞.

$$\chi_i(\infty) := \frac{1}{\omega_i^2(\lambda_{2i-1} + \lambda_{2i})^2} = \frac{1}{m_i^2}.$$

Note that for each $\lambda \in \overline{\mathbb{C}}$ the following equality holds.

$$\chi_i(\lambda) = \frac{a_{2i-1}a_{2i}}{m_i^2}. \tag{14}$$

5.1. Regularity

Let M be a regular stationary rotation (in the sense of definition 2). Find a basis such that J is diagonal and M is block-diagonal. Introduce the following subspaces:

- $K \subset \mathfrak{so}(n)$ is generated by $\{E_{2i-1,2i} - E_{2i,2i-1}\}_{i=1,\dots,m}$ and $\{E_{ij} - E_{ji}\}_{2m < i < j \leqslant n}$.

- $V_{ij} \subset \mathfrak{so}(n)$ is generated by $E_{2i-1,2j-1} - E_{2j-1,2i-1}$, $E_{2i-1,2j} - E_{2j,2i-1}$, $E_{2i,2j-1} - E_{2j-1,2i}$, $E_{2i,2j} - E_{2j,2i}$.
- $W_{ij} \subset \mathfrak{so}(n)$ is generated by $E_{2i-1,j} - E_{j,2i-1}$, $E_{2i,j} - E_{j,2i}$.

Clearly, the following vector space decomposition holds

$$\mathfrak{so}(n) = K \oplus \bigoplus_{1 \leqslant i < j \leqslant m} V_{ij} \oplus \bigoplus_{\substack{1 \leqslant i \leqslant m, \\ 2m < j \leqslant n}} W_{ij}. \tag{15}$$

Proposition 5.1. *The space K belongs to the common kernel of all brackets of the pencil at the point M. All spaces V_{ij}, W_i are mutually orthogonal with respect to all brackets of the pencil.*

Proof. Use (10). $\qquad\square$

Proposition 5.1 implies that the rank of a bracket P_λ drops if and only if this bracket is degenerate on one of the spaces V_{ij} or W_{ij}. Calculate the forms P_λ on these spaces.

Identify V_{ij} with the space of two-by-two matrices, and W_{ij} with $\mathbb{R}^2$. Let the matrices M_i be defined by (12). Let also the numbers a_j and the matrices A_i be defined by (13).

Proposition 5.2. *The form P_λ restricted to V_{ij} reads*

$$P_\lambda(X, Y) = 2\mathrm{Tr}\,(M_i X A_j Y^{\mathrm{t}} + M_j X^{\mathrm{t}} A_i Y).$$

The form P_λ restricted to W_{ij} reads

$$P_\lambda(v, w) = -2a_j v^{\mathrm{t}} M_i w.$$

Proof. Use (10). $\qquad\square$

Now calculate P_λ on V_{ij} in coordinates. Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_{ij}, \qquad Y = \begin{pmatrix} e & f \\ \mathfrak{g} & h \end{pmatrix} \in V_{ij}.$$

Explicit calculation shows that

$$P_\lambda(X, Y) = 2(m_i a_{2j-1} c + m_j a_{2i-1} b)e + 2(m_j a_{2i} d - m_i a_{2j-1} a)g$$
$$+ 2(m_i a_{2j} d - m_j a_{2i-1} a)f - 2(m_i a_{2j} b + m_j a_{2i} c)h.$$

Consequently, $X \in \mathrm{Ker}\, P_\lambda$ if and only if

$$\begin{cases} m_i a_{2j-1} c + m_j a_{2i-1} b = 0, \\ m_j a_{2i} d - m_i a_{2j-1} a = 0, \\ m_i a_{2j} d - m_j a_{2i-1} a = 0, \\ m_i a_{2j} b + m_j a_{2i} c = 0. \end{cases}$$

This system can be split into 2 two-by-two systems, and the determinant of both of them equals

$$\det = m_j^2 a_{2i-1} a_{2i} - m_i^2 a_{2j-1} a_{2j}.$$

So, the following is true.

Proposition 5.3. *P_λ is degenerate on V_{ij} if and only if*

$$m_j^2 a_{2i-1} a_{2i} - m_i^2 a_{2j-1} a_{2j} = 0. \tag{16}$$

If P_λ is degenerate on V_{ij}, then its kernel is given by

$$X = \begin{pmatrix} \alpha m_j a_{2i} & \beta m_i a_{2j-1} \\ -\beta m_j a_{2i-1} & \alpha m_i a_{2j-1} \end{pmatrix}, \tag{17}$$

where α and β are arbitrary numbers.

Now study P_λ on W_{ij}.

Proposition 5.4. *P_λ is degenerate (and, consequently, zero) on W_{ij} if and only if $\lambda = \lambda_j^2$.*

Proof. Use proposition 5.2.   □

Proposition 5.5. *The intersection of kernels of all brackets of the pencil is exactly K. For almost all brackets the kernel is exactly K. Thus, M is a regular equilibrium (in the sense of definition 7), and $\mathfrak{Ker}(M) = K$.*

Proof. Only finite numbers of brackets are degenerate on each of the spaces V_{ij} and W_{ij} (see propositions 5.3 and 5.4).   □

5.2. Description of the spectrum $\Lambda(M)$

Proposition 5.6. *Let M be a regular stationary rotation. Then $\Lambda(M)$ is the set of horizontal coordinates of intersection points in the parabolic diagram of M.*

Proof. By proposition 5.3, the bracket P_λ is degenerate on V_{ij} if and only if

$$m_j^2 a_{2i-1} a_{2i} - m_i^2 a_{2j-1} a_{2j} = 0.$$

This equality can be rewritten as (see (14))

$$\chi_i(\lambda) = \chi_j(\lambda),$$

which means that λ is the horizontal coordinate of the intersection point of two parabolas $y = \chi_i(x)$ and $y = \chi_j(x)$. Further, by proposition 5.4, P_λ is degenerate on W_{ij} if and only if $\lambda = \lambda_j^2$, which means that λ is the horizontal coordinate of the intersection point of the vertical line $x = \lambda_j^2$ with any parabola.   □

5.3. Description of adjoint operators

Compute the restriction of the operator $\mathrm{ad}_\lambda \, \mathrm{d}H_\lambda$ to the space $\mathrm{Ker}\, P_\lambda / K$. Using proposition 5.1, the kernel $\mathrm{Ker}\, P_\lambda(M)$ can be decomposed in the following way

$$\mathrm{Ker}\, P_\lambda = K \oplus \bigoplus_{1 \leqslant i < j \leqslant m} V_{ij}(\lambda) \oplus \bigoplus_{\substack{1 \leqslant i \leqslant m, \\ 2m < j \leqslant n}} W_{ij}(\lambda),$$

where

$$V_{ij}(\lambda) := \mathrm{Ker}\left(P_\lambda \mid_{V_{ij}} \right) \subset V_{ij}, \qquad W_{ij}(\lambda) := \mathrm{Ker}\left(P_\lambda \mid_{W_{ij}} \right) \subset W_{ij}. \qquad (18)$$

The space $\mathrm{Ker}\, P_\lambda / K$ is decomposed as

$$\mathrm{Ker}\, P_\lambda / K = \bigoplus_{1 \leqslant i < j \leqslant m} V_{ij}(\lambda) \oplus \bigoplus_{\substack{1 \leqslant i \leqslant m, \\ 2m < j \leqslant n}} W_{ij}(\lambda),$$

To compute $\mathrm{ad}_\lambda \, \mathrm{d}H_\lambda$, note that P_λ is a linear bracket, so the commutator in $\mathfrak{g}_\lambda$ is simply the restriction of the bracket $[\,,\,]_\lambda$ given by (8) to the space $\mathrm{Ker}\, P_\lambda$, so it is given by

$$[X, Y]_\lambda = XAY - YAX. \qquad (19)$$

Formula (11) implies that

$$\mathrm{d}H_\lambda = -(J + \sqrt{\lambda}E)^{-1}\Omega(J + \sqrt{\lambda}E)^{-1}, \tag{20}$$

so $\mathrm{d}H_\lambda$ is a block-diagonal matrix with two-by-two blocks on the diagonal. Denote the space of such matrices by L. Clearly, $L \subset K \subset \mathrm{Ker}\, P_\lambda$. Since $\mathrm{d}H_\lambda \in L$, it suffices to describe the operators $\mathrm{ad}_\lambda X$ for $X \in L$.

Proposition 5.7. *For any $X \in L$, the spaces $V_{ij}(\lambda)$ and $W_{ij}(\lambda)$ are invariant with respect to the operator* $\mathrm{ad}_\lambda X$.

Proof. Let $X \in L$ and $Y \in V_{ij}$. Then, using (19), show that $[X, Y]_\lambda \in V_{ij}$, which means that $\mathrm{ad}_\lambda X(V_{ij}) \subset V_{ij}$. Further, $V_{ij}(\lambda) = V_{ij} \cap \mathrm{Ker}\, P_\lambda$, and $\mathrm{Ker}\, P_\lambda$ is invariant with respect to $\mathrm{ad}_\lambda X$, so $V_{ij}(\lambda)$ is invariant as the intersection of two invariant subspaces. The proof for $W_{ij}(\lambda)$ is the same. $\square$

Represent an element $X \in L$ as

$$X = \begin{pmatrix} 0 & x_1 & \\ -x_1 & 0 & \\ & & \ddots \end{pmatrix}.$$

Proposition 5.8. *Let $V_{ij}(\lambda) \neq 0$. Then the operator* $\mathrm{ad}_\lambda X$*, being written in coordinates α, β given by (17), reads*

$$\begin{pmatrix} 0 & -a_{2i-1}\left(x_i - \dfrac{m_j}{m_i}x_j\right) \\ a_{2i}\left(x_i - \dfrac{m_j}{m_i}x_j\right) & 0 \end{pmatrix}.$$

Proof. Use (19). $\square$

Proposition 5.9. *Let $W_{ij}(\lambda) \neq 0$. Then, after the natural identification of $W_{ij}(\lambda) = W_{ij}$ with $\mathbb{R}^2$, the matrix of* $\mathrm{ad}_\lambda X$ *reads*

$$\begin{pmatrix} 0 & x_i a_{2i} \\ -x_i a_{2i-1} & 0 \end{pmatrix}.$$

Proof. Use (19). $\square$

Proposition 5.10.

(1) Let $V_{ij}(\lambda) \neq 0$. Then the eigenvalues of $\mathrm{ad}_\lambda X$ *restricted to $V_{ij}(\lambda)$ are $\pm v_{ij}(X)$, where*

$$v_{ij}(X) = \sqrt{-\chi_i(\lambda)}(m_j x_j - m_i x_i).$$

(2) Let $W_{ij}(\lambda) \neq 0$. Then the eigenvalues of $\mathrm{ad}_\lambda X$ *restricted to $W_{ij}(\lambda)$ are $\pm\mu_i(X)$, where*

$$\mu_i(X) = \sqrt{-\chi_i(\lambda)}m_i x_i.$$

Proof. Use propositions 5.8 and 5.9. $\square$

5.4. Spectrum of the linearized system

Let M be a regular stationary rotation. Then the symplectic leaves of generic brackets $P_\alpha, \alpha \notin \Lambda(M)$ are tangent at M. By $T(M)$ denote their common tangent space. By dX denote the linearization of (3) at M.

Proposition 5.11. *Let M be a regular stationary rotation. Let also $\lambda_{ij}^{(1)}, \lambda_{ij}^{(2)}$ be two roots of the equation $\chi_i(\lambda) = \chi_j(\lambda)$. Then the spectrum of $dX \mid_{T(M)}$ is*

$$\sigma(dX \mid_{T(M)}) = \{\pm\sigma_{ij}^{(k)}\}_{1\leqslant i<j\leqslant m}^{k=1,2} \cup \{\pm\tau_{ij}\}_{1\leqslant i\leqslant m}^{2m+1\leqslant j\leqslant n}$$

where

$$\sigma_{ij}^{(k)} = \frac{1}{\sqrt{-\chi_i\left(\lambda_{ij}^{(k)}\right)}}\left(\frac{\lambda_{ij}^{(k)} + \lambda_{2i-1}\lambda_{2i}}{\lambda_{2i-1} + \lambda_{2i}} - \frac{\lambda_{ij}^{(k)} + \lambda_{2j-1}\lambda_{2j}}{\lambda_{2j-1} + \lambda_{2j}}\right) \qquad if\ \lambda_{ij}^{(k)} \neq \infty,$$

$$\sigma_{ij}^{(k)} = (\omega_j - \omega_i)\sqrt{-1} \qquad if\ \lambda_{ij}^{(k)} = \infty,$$

and

$$\tau_{ij} = \frac{1}{\sqrt{-\chi_i\left(\lambda_j^2\right)}}\frac{(\lambda_j - \lambda_{2i-1})(\lambda_j - \lambda_{2i})}{\lambda_{2i-1} + \lambda_{2i}}.$$

Proof. Use lemma 3.1, proposition 5.10 and formula (20). □

Remark 5.1. It is also possible to find the spectrum of the linearized system explicitly, without introducing the bi-Hamiltonian structure. However, the bi-Hamiltonian framework is essential for the proof of the stability part of theorem 1 (section 6), so it seems to be better to prove both parts using the same philosophy. At the same time, the bi-Hamiltonian approach is simpler from the computational viewpoint. Also note that lemma 3.1 allows to find the spectrum for *all* systems bi-Hamiltonian with respect to Π *at once*.

5.5. Completion of the proof

For simplicity assume that all eigenvalues of Ω are distinct. Suppose that there is at least one intersection in the parabolic diagram which is either complex or belongs to the lower half-plane. Then proposition 5.11 shows that dX has an eigenvalue with a non-zero real part unless

$$\frac{\lambda_{ij}^{(k)} + \lambda_{2i-1}\lambda_{2i}}{\lambda_{2i-1} + \lambda_{2i}} - \frac{\lambda_{ij}^{(k)} + \lambda_{2j-1}\lambda_{2j}}{\lambda_{2j-1} + \lambda_{2j}} = 0. \tag{21}$$

A simple computation shows that (21) implies the equality $\omega_i^2 = \omega_j^2$. If all eigenvalues of Ω are distinct, then this is not possible, so dX has an eigenvalue with a non-zero real part, and the equilibrium is unstable.

The sketch of the proof in the case when Ω has multiple eigenvalues is as follows. Assume that (21) is satisfied. Then $\lambda_{ij}^{(k)}$ is a real number. So, we only need to consider the case when there is a real intersection in the lower half-plane. Find a stationary rotation $M_\varepsilon \in U_\varepsilon(M)$ such that all eigenvalues of $\Omega(M_\varepsilon)$ are distinct. Then the rotation M_ε is unstable. Moreover, an argument similar to the one of [3] can be used to show that there is a heteroclinic trajectory joining M_ε with another stationary rotation M_ε^- such that the distance $\text{dist}(M_\varepsilon, M_\varepsilon^-)$ is uniformly bounded from below as $\varepsilon \to 0$. Therefore M is unstable.

6. Proof of theorem 1: stability

According to theorem 4, to prove the stability part of theorem 1 we should do the following:

(1) check that the pencil Π is fine at M (section 6.1);
(2) check that the equilibrium point M is strongly regular (section 6.2);
(3) check that the spectrum of Π at M is real (section 6.3);
(4) check that the pencil Π is diagonalizable at x (section 6.4);
(5) check that for each $\lambda \in \Lambda_\Pi(x)$ the λ-linearization $d_\lambda \Pi(x)$ is compact (section 6.5).

6.1. The pencil Π is fine

Let α be such that $\alpha \notin \Lambda(M)$ and $\alpha < \lambda_{\min}^2$ where $\lambda_{\min}$ is the minimal eigenvalue of J. Take ε such that $\alpha + \varepsilon < \lambda_{\min}^2$, and $(\alpha - \varepsilon, \alpha + \varepsilon) \cap \Lambda(M) = \varnothing$. Take $U = (\alpha - \varepsilon, \alpha + \varepsilon)$. Then for each $\beta \in U$, the bracket P_β is compact semisimple. Therefore conditions 1 and 2 of definition 13 are satisfied. Further, for any $\beta \in U$, the map

$$F_{\alpha\beta} : (\mathfrak{so}(n), [\,,\,]_\alpha) \to (\mathfrak{so}(n), [\,,\,]_\beta)$$

defined by

$$F_{\alpha\beta}(X) = (J^2 - \beta E)^{-1/2}(J^2 - \alpha E)^{1/2} X (J^2 - \alpha E)^{1/2}(J^2 - \beta E)^{-1/2}$$

is an isomorphism of Lie algebras. Therefore, for any $f_\alpha \in \mathcal{Z}(P_\alpha)$, the function

$$f_\beta(x) = f_\alpha(F_{\alpha\beta}^*(x))$$

is a Casimir function of P_β, so condition 3 of definition 13 is also satisfied, and the pencil is fine at every point.

6.2. Strong regularity

Proposition 5.5 shows that a regular stationary rotation (in the sense of definition 2) is a regular equilibrium (in the sense of definition 7). Now, prove that each regular stationary rotation is strongly regular (see definition 12). Introduce the following subspaces:

- K_0 is generated by $\{E_{2i-1,2i} - E_{2i,2i-1}\}_{i=1,\dots,m}$;
- K_1 is generated by $\{E_{ij} - E_{ji}\}_{2m < i < j \leqslant n}$.

Then

$$K = K_0 \oplus K_1$$

as a Lie pencil, which means that K_0 and K_1 are Lie subalgebras with respect to all Lie structures $[\,,\,]_\alpha$, and $[K_0, K_1]_\alpha = 0$.

Clearly, K_0 is Abelian with respect to all structures $[\,,\,]_\alpha$, and K_1 is isomorphic to $\mathfrak{so}(n)$ with a Lie pencil given by

$$[X, Y]_\lambda = [X, Y]_0 - \lambda[X, Y]_\infty = X(J_1^2 - \lambda E)Y - Y(J_1^2 - \lambda E)X,$$

where $J_1 = \mathrm{diag}(\lambda_{2m+1}, \dots, \lambda_n)$. Therefore the centre of K_1 with respect to any Lie structure $[\,,\,]_\alpha$ is trivial unless $n - 2m = 2$. So, $\mathcal{Z}_\alpha(K) = K_0$ for all α if $n - 2m \neq 2$, and $\mathcal{Z}_\alpha(K) = K$ if $n - 2m = 2$. In both cases $\mathcal{Z}_\alpha(K)$ does not depend on α, so M is strongly regular.

6.3. The spectrum $\Lambda(M)$ is real

By proposition 5.6, the spectrum $\Lambda(M)$ is the set of horizontal coordinates of the intersection points on the parabolic diagram of M. So, under the conditions of theorem 1, the spectrum is real.

6.4. Diagonalizability

It is convenient to use the following alternative definition of diagonalizability.

Proposition 6.1. *Assume that x is a regular equilibrium of a bi-Hamiltonian system, and that the spectrum $\Lambda_\Pi(x)$ is real. Then the pencil Π is diagonalizable at the point x if and only if*

$$\mathrm{T}_x^* M / \mathfrak{Ker}(x) = \bigoplus_{\lambda \in \Lambda(x)} \mathrm{Ker}\, P_\lambda(x) / \mathfrak{Ker}(x).$$

For the proof, see [22, 23].

Proposition 6.2. *Let M be a regular stationary rotation. Then the pencil is diagonalizable at M if and only if any two parabolas in the parabolic diagram of M intersect at two different points.*

Proof. Proposition 6.1 implies that the pencil is diagonalizable if and only if

$$\mathfrak{so}(n)/K = \bigoplus_{\lambda \in \Lambda(M)} \mathrm{Ker}\, (P_\lambda)/K. \tag{22}$$

Using (15), write

$$\mathfrak{so}(n)/K = \bigoplus_{1 \leqslant i < j \leqslant m} V_{ij} \oplus \bigoplus_{\substack{1 \leqslant i \leqslant m, \\ 2m < j \leqslant n}} W_{ij}.$$

Since all the summands of this decomposition are pairwise orthogonal with respect to P_λ (proposition 5.1), relation (22) is satisfied if and only if

$$V_{ij} = \bigoplus_{\lambda \in \Lambda(M)} V_{ij}(\lambda) \qquad \text{for } 1 \leqslant i < j \leqslant m, \tag{23}$$

$$W_{ij} = \bigoplus_{\lambda \in \Lambda(M)} W_{ij}(\lambda) \qquad \text{for } 1 \leqslant i \leqslant m, 2m < j \leqslant n, \tag{24}$$

where $V_{ij}(\lambda)$ and $W_{ij}(\lambda)$ are defined by (18).

Since there is a unique $\lambda = \lambda_j^2$ such that $W_{ij} = W_{ij}(\lambda)$, equality (24) is always satisfied. Equality (23) is satisfied if and only if equation (16) has two distinct roots, i.e. if corresponding two parabolas are not tangent to each other. $\qquad\square$

6.5. Compactness

Show that under the conditions of theorem 1, the pencil $\mathrm{d}_\lambda \Pi(M)$ is compact.

First, consider the case $\lambda = \infty$. Then $\mathfrak{g}_\lambda$ is the ad^* stabilizer of $M \in \mathfrak{so}(n)^*$, so $\mathfrak{g}_\lambda$ is compact, and so is the pencil $\mathrm{d}_\infty \Pi(M)$ (see proposition 3.4).

So, let $\lambda \neq \infty$. The pencil $\mathrm{d}_\lambda \Pi(M)$ is defined on $\mathfrak{g}_\lambda$ by the cocycle $\mathcal{B} = P_\infty \mid_{\mathrm{Ker}\, P_\lambda}$. Prove that there exists $X \in \mathcal{Z}(\mathrm{Ker}\, \mathcal{B})$ such that the form

$$\mathfrak{B}_X(Y, Y) := P_\infty([X, Y]_\lambda, Y)$$

is positive definite on $\mathrm{Ker}\, P_\lambda / \mathrm{Ker}\, \mathcal{B}$.

By proposition 6.2, the pencil is diagonalizable at M. This implies that (see definition 11)

$$\dim \mathrm{Ker}\, (P_\infty \mid_{\mathrm{Ker}\, P_\lambda}) = \dim \mathrm{Ker}\, P_\infty.$$

Since $\mathrm{Ker}\, P_\infty = K$ (proposition 5.5),

$$\dim \mathrm{Ker}\, (P_\infty \mid_{\mathrm{Ker}\, P_\lambda}) = \dim K.$$

On the other hand, $\mathrm{Ker}\, (P_\infty \mid_{\mathrm{Ker}\, P_\lambda}) \supset K$. So,

$$\mathrm{Ker}\, \mathcal{B} = \mathrm{Ker}\, (P_\infty \mid_{\mathrm{Ker}\, P_\lambda}) = K, \tag{25}$$

and the compactness condition can be reformulated as follows: there exists $X \in \mathcal{Z}(K)$ such that the form $\mathcal{B}_X$ is positive definite on

$$\operatorname{Ker} P_\lambda / K = \bigoplus_{1 \leqslant i < j \leqslant m} V_{ij}(\lambda) \oplus \bigoplus_{\substack{1 \leqslant i \leqslant m, \\ 2m < j \leqslant n}} W_{ij}(\lambda).$$

At the same time, $\mathcal{Z}(K) = K_0$ or $\mathcal{Z}(K) = K$ (section 6.2), so $K_0 \subset \mathcal{Z}(K)$, and it suffices to show that there exists $X \in K_0$ such that $\mathcal{B}_X$ is positive on $\operatorname{Ker} P_\lambda / K$. Represent an element $X \in K_0$ as

$$X = \begin{pmatrix} 0 & x_1 & & & & & \\ -x_1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & x_m & & \\ & & & -x_m & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix}.$$

Denote

$$b_i := a_{2i} + a_{2i-1}.$$

Proposition 6.3. *Let* $V_{ij}(\lambda) \neq 0$,

$$Y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in V_{ij}(\lambda), \qquad Z = \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{pmatrix} \in V_{ij}(\lambda),$$

where α, β *are the coordinates on* $V_{ij}(\lambda)$ *given by (17). Then*

$$P_\infty(Y, Z) = 2m_i a_{2j-1} \left(m_i^2 b_j - m_j^2 b_i \right) (\alpha \widetilde{\beta} - \widetilde{\alpha} \beta).$$

Proof. Use proposition 5.2. □

Proposition 6.4. *Let*

$$Y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in V_{ij}(\lambda) \neq 0,$$

where α, β *are the coordinates on* $V_{ij}(\lambda)$ *given by (17). Then*

$$\mathcal{B}_X(Y, Y) = -2a_{2j-1} \left(m_i^2 b_j - m_j^2 b_i \right) (m_i x_i - m_j x_j)(a_{2i}\alpha^2 + a_{2i-1}\beta^2). \quad (26)$$

Proof. Use propositions 6.3 and 5.8. □

Proposition 6.5. *Let*

$$Y = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in W_{ij}(\lambda) \neq 0,$$

where α, β *are the coordinates on* $W_{ij}(\lambda) = W_{ij}$ *given by its natural identification with* $\mathbb{R}^2$. *Then*

$$\mathcal{B}_X(Y, Y) = -2m_i x_i (a_{2i-1}\alpha^2 + a_{2i}\beta^2). \quad (27)$$

Proof. Use propositions 5.2 and 5.9. □

Proposition 6.6. *Under the conditions of theorem 1, the pencil* $d_\lambda \Pi(M)$ *is compact.*

Proof. It suffices to consider the case $\lambda \neq \infty$ (see beginning of the section). Show that there exists $X \in K_0$ such that $\mathcal{B}_X$ is positive on

$$\mathrm{Ker}\, P_\lambda / K = \bigoplus V_{ij}(\lambda) \oplus \bigoplus W_{ij}(\lambda).$$

The summands of this decomposition are pairwise orthogonal with respect to $\mathcal{B}_X$, so it suffices to show that there exists X such that $\mathcal{B}_X$ is positive-definite on each of the summands. Propositions 6.4 and 6.5 show that *for each summand there exists X such that $\mathcal{B}_X$ is positive on this summand* (note that a_{2i} and a_{2i-1} are of the same sign; this is because all intersections are in the upper half-plane). However, it is not obvious *a priori* why there should exist X such that $\mathcal{B}_X$ is positive on *all summands*. Nevertheless, such an X exists and can be defined by the following magical formula

$$x_i := -\frac{m_i}{b_i} \tag{28}$$

for $i = 1, \ldots, m$. Show that $\mathcal{B}_X > 0$ on $V_{ij}(\lambda)$. Substituting (28) into (26), obtain

$$\mathcal{B}_X(Y, Y) = 2a_{2j-1}b_j(a_{2i}\alpha^2 + a_{2i-1}\beta^2)b_i \left(\frac{m_i^2}{b_i} - \frac{m_j^2}{b_j} \right)^2.$$

Since all intersections are in the upper half-plane, a_{2j-1} and b_j have the same sign. The same is true for a_{2i}, a_{2i-1} and b_i. Consequently, for $Y \neq 0$, the following inequality is satisfied

$$a_{2j-1}b_j(a_{2i}\alpha^2 + a_{2i-1}\beta^2)b_i > 0.$$

Further, if

$$\frac{m_i^2}{b_i} - \frac{m_j^2}{b_j} = 0,$$

then $P_\infty(Y, Y) = 0$ (see proposition 6.3), and

$$V_{ij}(\lambda) \subset \mathrm{Ker}\left(P_\infty \mid_{\mathrm{Ker}\, P_\lambda} \right),$$

which contradicts (25). So, $\mathcal{B}_X$ is positive on $V_{ij}(\lambda)$.

Now, show that $\mathcal{B}_X > 0$ on $W_{ij}(\lambda)$. Substituting (28) into (26), obtain

$$\mathcal{B}_X(Y, Y) = 2\frac{m_i^2}{b_i}(a_{2i-1}\alpha^2 + a_{2i}\beta^2).$$

Since a_{2i}, a_{2i-1} and b_i are of the same sign, and $m_i \neq 0$, the form $\mathcal{B}_X > 0$ on $W_{ij}(\lambda)$. $\square$

Remark 6.1. Let $\lambda > \lambda_{\max}^2$ or $\lambda < \lambda_{\min}^2$ where $\lambda_{\min}$ and $\lambda_{\max}$ are, respectively, the minimal and the maximal eigenvalues of J. Then the compactness of $\mathrm{d}_\lambda \Pi(M)$ is natural, since the algebra $\mathfrak{so}(n)$ with the $[\,,\,]_\lambda$ bracket is compact, and so is $\mathfrak{g}_\lambda(M)$ which is the ad_λ^* stabilizer of M.

However, for $\lambda \in [\lambda_{\min}^2, \lambda_{\max}^2]$, the algebra $\mathfrak{g}_\lambda(M)$ is not necessarily compact. The classification of Lie algebras $\mathfrak{g}_\lambda(M)$ up to an isomorphism is given in the appendix.

Acknowledgments

Appendix: Classification of Lie algebras $\mathfrak{g}_\lambda$

Let M be a regular stationary rotation, $\lambda \in \overline{\mathbb{C}}$. By Σ_λ denote the set of intersection points on the parabolic diagram of M with abscissa $x = \lambda$. For $\lambda \in \mathbb{R}$, denote by Σ_λ^+, Σ_λ^- the set
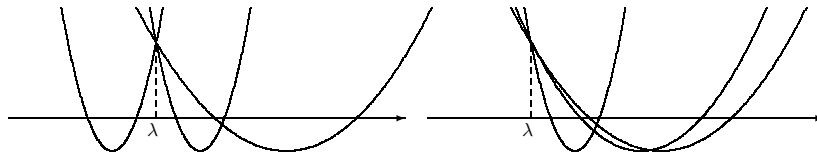
Figure 7. Rotations with $\mathfrak{g}_\lambda \simeq \mathfrak{u}(1, 2) \oplus \mathbb{R}^N$ and $\mathfrak{g}_\lambda \simeq \mathfrak{u}(3) \oplus \mathbb{R}^N$, respectively.

of intersection points $\in \Sigma_\lambda$ which lie in the upper and lower half-planes, respectively. For $z \in \Sigma_\lambda$ denote by n_z the number of parabolas passing through z. For $z \in \Sigma_\lambda^+$ denote by l_z, r_z the number of parabolas passing through z such that their vertices are to the left or right of z, respectively. Denote by v the number of vertical lines on the parabolic diagram. For $\lambda \in \mathbb{R}$, denote by l_λ and r_λ the number of vertical lines to the left or to the right of the line $x = \lambda$, respectively.

Proposition A.1. *Let M be a regular stationary rotation.*

(1) If $\lambda \in \mathbb{R}$, and there is no vertical line $x = \lambda$ on the parabolic diagram of M, then

$$\mathfrak{g}_\lambda \simeq \mathfrak{so}(l_\lambda, r_\lambda) \oplus \bigoplus_{z \in \Sigma_\lambda^+} \mathfrak{u}(l_z, r_z) \oplus \bigoplus_{z \in \Sigma_\lambda^-} \mathfrak{gl}(n_z, \mathbb{R}) \oplus \mathbb{R}^N.$$

(2) If $\lambda = \infty$, then

$$\mathfrak{g}_\lambda \simeq \mathfrak{so}(v, \mathbb{R}) \oplus \bigoplus_{z \in \Sigma_\lambda} \mathfrak{u}(n_z, \mathbb{R}) \oplus \mathbb{R}^N.$$

(3) If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then

$$\mathfrak{g}_\lambda \simeq \mathfrak{so}(v, \mathbb{C}) \oplus \bigoplus_{z \in \Sigma_\lambda} \mathfrak{gl}(n_z, \mathbb{C}) \oplus \mathbb{C}^N.$$

(4) If $\lambda \in \mathbb{R}$, and there is a vertical line $x = \lambda$ on the parabolic diagram of M, then

$$\mathfrak{g}_\lambda \simeq (\mathfrak{so}(l_\lambda, r_\lambda) \ltimes_{\rho_1} \mathbb{R}^{l_\lambda + r_\lambda}) \oplus \bigoplus_{z \in \Sigma_\lambda^+} (\mathfrak{u}(l_z, r_z) \ltimes_{\rho_2} \mathbb{C}^{l_z + r_z}) \oplus \bigoplus_{z \in \Sigma_\lambda^-} (\mathfrak{gl}(n_z, \mathbb{R}) \ltimes_{\rho_3} \mathbb{R}^{2n_z}) \oplus \mathbb{R}^N,$$

where representations ρ_1, ρ_2 are standard, and

$$\rho_3(A) = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.$$

In all cases N is some number $\geqslant 0$.

Example A.1. Rotations with $\mathfrak{g}_\lambda \simeq \mathfrak{u}(1, 2) \oplus \mathbb{R}^N$ and $\mathfrak{g}_\lambda \simeq \mathfrak{u}(3) \oplus \mathbb{R}^N$, respectively are depicted in figure 7. Proposition 5.11 can be used to check that both cases correspond to a $(1 : 1 : 1)$ resonance. Note that, in both cases, the bi-Hamiltonian system corresponding to the linear pencil $d_\lambda \Pi$ coincides with the three-wave interaction system [34]. So, the three-wave interaction system is the bi-Hamiltonian linearization of the multidimensional rigid body at a $(1 : 1 : 1)$ resonance.

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