

Making Sense of Explaining Why

Adapted from Sybilla Beckmann

A Good Explanation

Question: Use the *meaning of fractions* to explain why

$$\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$$

(In other words, explain why $\frac{2}{3} = \frac{114}{171}$.) *Do not* use multiplying by 1 to explain this.

Solution: Using the guidelines that are found on the next page, we could respond to the problem “using the meaning of fractions to explain why $\frac{2}{3} = \frac{2 \cdot 57}{3 \cdot 57}$ ” as follows:

According to the meaning of fractions, $\frac{2}{3}$ of a pie is the amount formed by 2 parts when the pie is divided into 3 equal parts. This amount is shown shaded in the picture below. [Show the relevant picture of a pie.] If I divide each of those 3 equal parts into 57 small equal parts, the pie will now be divided into $3 \cdot 57 = 171$ small parts. Because the 2 original shaded parts representing $\frac{2}{3}$ of the pie have each been subdivided into 57 small parts, these 2 original shaded parts become $2 \cdot 57 = 114$ small parts, as indicated in the picture. [Show another picture of the same pie, indicating that each piece is now subdivided into many smaller pieces of equal size.] It’s still the same amount of pie that is shaded either way you look at it. So 2 of the original 3 parts of pie is the same amount of pie as $2 \cdot 57$ small parts of the total $3 \cdot 57$ small parts. This is why $\frac{2}{3}$ of a pie is the same amount of pie as $\frac{2 \cdot 57}{3 \cdot 57} = \frac{114}{171}$ of the pie.

A Bad Explanation

Question: Susan was supposed to use $\frac{5}{4}$ of a cup of butter in her recipe but she only used $\frac{3}{4}$ of a cup of butter. What fraction of the butter that she should have used did Susan actually use? Draw pictures to help you solve this problem. Explain your answer clearly. For each fraction in this problem, and in your solution, describe the *whole* with which the fraction is associated.

Solution: One student responded by drawing pictures to show $\frac{5}{4}$ and $\frac{3}{4}$ cups of butter, and then calculated:

$$\frac{3}{4} \div \frac{5}{4} \text{ or } \frac{3}{4} \cdot \frac{4}{5} = \frac{12}{20} = \frac{6}{10} = \frac{3}{5} \text{ out of } \frac{5}{4}.$$

She went on to explain as follows:

... To find the fraction of the butter that Susan used out of what she should have used you need to divide $\frac{3}{4}$ and $\frac{5}{4}$. When dividing fractions you can take the reciprocal of the second fraction and multiply it by the first fraction: $\frac{3}{4} \cdot \frac{5}{4}$. When you do that you find that Susan used $\frac{3}{5}$ of the $\frac{5}{4}$ of butter ...

Despite the directions to use a picture to help solve the problem, the student showed (correct) calculations and discussed those calculations.

Guidelines that Characterize Good Explanations in Mathematics

1. The explanation is factually correct, or nearly so, with only minor flaws (for example, a minor mistake in a calculation).
2. The explanation addresses the specific question or problem that was posed. It is focused, detailed, and precise. There are no irrelevant or distracting points.
3. The explanation is clear, convincing, and logical. A clear and convincing explanation is characterized by the following:
 - (a) The explanation can be used to teach another (college) student, possibly even one who is not in the class.
 - (b) The explanation could be used to convince a skeptic.
 - (c) The explanation does not require the reader to make a leap of faith.
 - (d) Key points are emphasized.
 - (e) If applicable, supporting pictures, diagrams and/or equations are used appropriately and as needed.
 - (f) The explanation is coherent.
 - (g) Clear, complete sentences are used.