Lecture 2: Existence Theorem for Minimizers of Lagrangian Type Functionals

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March 28, 2010

1 Introduction

In this lecture we present a theorem that guarantees the existence of minimizers of a Lagrangian type functional $I : W^{1,q} \to \mathbb{R}$. The conditions of the theorem are general and can be extended to other variational type problems.

2 Notation

We will use the following notation and symbols throughout this lecture.

1. $\Omega \subset \mathbb{R}^n$ is a bounded, open set, with C^1 boundary.
2. $L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ is a smooth mapping called the **Lagrangian**.
3. The energy functional is defined by
$$I[u] = \int_\Omega L(Du, u, x)\, dx.$$
4. g is a smooth function defined on $\partial\Omega$.
5. The **admissible set** is the family of functions
$$\mathcal{A} = \{u \in W^{1,q} : u = g \text{ on } \partial\Omega\}.$$

In this lecture we are interested in studying the following variational problem:

Problem 1. *Find sufficient conditions to guarantee that there exists $u^* \in \mathcal{A}$ such that u^* minimizes $I[\cdot]$.*

Finally, we will assume the Einstein convention of summation over repeated indices unless stated otherwise.

3 Motivation

Let $f(x)$ be a function with domain $\mathbb{R}$. In calculus we learn that a necessary condition for the function to have a minimizer is that the first derivative vanishes at the potential minimum value. Although the Euler-Lagrange equations are an analogue of this condition, in general these equations cannot be solved analytically. Therefore, we will look for weaker conditions that will allow us to show that a minimizer exists.

To illustrate what conditions we will need, consider the following function

$$f(x) = \frac{1}{1+x^2}.$$

This function is bounded below but it does not obtain a minimum value. Intuitively, what we need to be able to do is control $f(x)$ for large values of $|x|$ and have some form of continuity. It turns out that we do not need full continuity (this is nice because in the infinite dimensional setting our functionals are rarely continuous).

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***lower semi-continuous*** *if $x_n \to x$ in $\mathbb{R}$ implies that*

$$f(x) \leq \liminf_{n\to\infty} f(x_k)$$

Proposition 1. *Let $f(x)$ be a lower semi-continuous function on $\mathbb{R}$. If $f(x) \geq 0$ and $f(x) \to \infty$ as $|x| \to \infty$, then $f(x)$ has a minimizer.*

Proof. Let $m = \inf_x \{f(x)\}$. Then, there exists a sequence x_n such that $f(x_n) \to m$. Therefore, $\forall n \in \mathbb{N}$, $\exists C \in \mathbb{R}$ such that $f(x_n) < C$. Now, since $\lim_{|x|\to\infty} = \infty$ it follows that $\exists M, w \in \mathbb{R}$ such that if $|x| > w$ then $f(x) > M$. Therefore, $x_n \in [-w, w]$ and consequently has a convergent subsequence x_{n_k} with limit x^*. Thus, we can conclude from semi-continuity that

$$m = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) \geq \liminf_{k\to\infty} f(x_{n_k}) \geq f(x^*).$$

This proves that x^* is the minimizer. □

The key analytical tools we used in this proof were compactness and semi-continuity. The main problem with extending these ideas to Banach spaces is that the unit ball is not necessarily compact.

Theorem 3.1. *Let X be an infinite dimensional normed linear space; than the unit ball is not compact*

Proof. We first require a lemma about any normed linear space, not just infinite dimensional.

Lemma 3.1. *Let Y be a closed, proper subspace of the normed linear space X. Then there is a vector z in X satisfying*

$$|z - y| > 1/2 \text{ for all } y \text{ in } Y \text{ and } \|z\| = 1.$$

Proof. Since Y is a proper subspace of X, $\exists x \in X$ such that $x \notin Y$ and since Y is closed x has a positive distance from Y:

$$\inf_{y \in Y} ||x - y|| = d > 0.$$

There is a $y \in Y$ such ath

$$||x - y_0|| < 2d.$$

Let $z = x - y_0$ and note that $||z|| < 2d$. Now, by linearity it follows that $\forall y \in Y$

$$||z - y|| = ||x - y_0 - y|| \geq d.$$

Therefore if we normalize z to obtain $z' = \frac{z}{||z||}$ we have that $\forall y \in Y$

$$||z' - y|| = \left\| \frac{z}{||z||} - y \right\| = \left\| \frac{z - y}{||z||} \right\| > \frac{d}{2d} = \frac{1}{2}$$

□

We now return to the proof of the theorem. Let $y_1 \in Y$ such that $||y_1|| = 1$ and we will construct a sequence recursively as follows. Suppose $y_1, \ldots, y_{n-1}$ have been chose. Then, $Y_n = \text{span}\{y_1, \ldots, y_{n-1}\}$ is a proper subspace of X and is furthermore closed since it is finite dimensional. Therefore, we can select y_n such that $||y_n|| = 1$ and $\forall j < n$, $||y_n - y_j|| > \frac{1}{2}$. Consequently, any subsequence of y_n will fail to be Cauchy sequence and therefore $\{y_n\}$ has no convergent subsequence. □

4 Coercivity Condition

In this section we will extend the simple theorem about functions to Lagrangian type functionals on the Banach space $W^{1,q}(\Omega)$. Let $u_k \in W^{1,q}$. Then, $||u_k||_{W^{1,q}} \to \infty$ if and only if $||Du_k||_{L^q} \to \infty$. This motivates the following definition.

Definition 2. *L is **coercive** if there exists $\alpha, \beta > 0$ such that $\forall p \in \mathbb{R}^n, z \in \mathbb{R}$, and $x \in \Omega$*

$$L(p, z, x) \geq \alpha |p|^q - \beta. \tag{4.1}$$

Definition 3. *Let $u_k \in \mathcal{A}$, be a sequence such that $I[u_k] \to \inf_{u \in \mathcal{A}} I[u]$. We call u_k a **minimizing sequence**.*

Lemma 4.1. *If L is coercive and $I[\cdot]$ is bounded then any minimizing sequence of $I[\cdot]$ lies in a bounded subset of $W^{1,q}(\Omega)$.*

Unfortunately, since the unit ball is not compact in $W^{1,q}$ we can not say that the limit point of the minimizing sequence lies in $W^{1,q}$. But, the unit ball is weakly compact.

Theorem 4.1. *The closed unit ball in a Banach space X is compact in the weak topology if and only X is reflexive ($(X^*)^* = X$).*

Definition 4. *A function $I[\cdot]$ is weakly lower semi-continuous if*

$$I[u] \leq \liminf_{k \to \infty} I[u_k]$$

whenever $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$.

Corollary 4.1. *If a Largrangian type functional is bounded below, coercive, and weakly lower semi-continuous then it has a minimizer.*

The final theorem gives us an easier method to check if a functional is weakly lower semi-continuous.

Theorem 4.2. *Assume L is smooth, positive, and the mapping $p \mapsto L(p, z, x)$ is convex for each $z \in \mathbb{R}$ and $x \in \Omega$. Then, $I[\cdot]$ is weakly lower semi-continuous in $W^{1,q}(\Omega)$.*

Proof. Let u_k be a minimizing sequence such that $u_k \rightharpoonup u$ and let $l = \liminf_{k\to\infty}\{u_k\}$. Therefore, u_k is bounded and by taking a subsequence we can assume $lim_{k\to\infty} I[u_k] = l$. Recall that $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ meaning that each bounded sequence in $W^{1,p}(\Omega)$ is pre-compact in $L^p(\Omega)$. Therefore, taking another subsequence we have that $u_k \to u$ a.e. in $L^q(\Omega)$.

Fix $\epsilon > 0$. Then Egoroff's theorem asserts that $u_k \to u$ uniformly on E_ϵ, where $|\Omega - E_\epsilon| < \epsilon$. Let $F_\epsilon = \{x \in \Omega : |u(x)| + |Du(x)| < \frac{1}{\epsilon}\}$. Let $G_\epsilon = E_\epsilon \cap F_\epsilon$.

Then,

$$I[u_k] = \int_\Omega L(Du_k, u_k, x)\,dx \geq \int_{G_\epsilon} L(Du_k, u_k, x)\,dx.$$

Therefore, from convexity,

$$I[u_k] \geq \int_{G_\epsilon} L(Du, u_k, x)\,dx + \int_{G_\epsilon} D_p L(Du, u_k, x) \cdot (Du_k - Du)\,dx.$$

Consequently,

$$\lim_{k\to\infty} I[u_k] \geq \int_{G_\epsilon} L(Du, u, x)\,dx.$$

Finally, by the monotone convergence theorem we have that

$$l \geq I[u].$$

□