Lecture 3: Elliptic Operators

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1 Introduction

Let L be an operator of the form

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where α is a multiple index, $x \in \mathbb{R}^d$. That is $\alpha = (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \mathbb{N}$, and D is the differential operator,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}},$$

with $|\alpha| = \alpha_1 + ... + \alpha_d$. We suppose that functions $a_\alpha(x) \in C^\infty(\mathbb{R}^d)$.

Definition 1. *We define an operator L as elliptic at a point $x \in \mathbb{R}^d$ if*

$$L_p(\xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \neq 0, \quad \textit{for all } \xi \neq 0 \textit{ in } \mathbb{R}^d.$$

Definition 2. *An operator L is strongly elliptic (or uniformly) if*

$$L_p(\xi) \geq \epsilon \|\xi\|,$$

for some $\epsilon > 0$.

What is useful about elliptic partial differential equations?

1.1 Regularity:

1. Let L be elliptic in $\mathbb{R}^d$ and satisfy $Lf = g$ where $g \in C^\infty$. Then by ellipticity of L, $f \in C^\infty$.

2. Under the same conditions with $a_\alpha(x)$ begin real analytic $Lf = g$ with g real analytic. Then f is also real analytic.

1.2 An example of the importance of real analyticity:

Let $f(x)$ be defined as

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Note that f is C^∞, but has no Taylor series expansion at $x = 0$. Therefore f is not real analytic.

2 Motivation

Consider the differential equation

$$\begin{aligned} \tfrac{dx_t}{dt} &= f(t, x_t) \\ x_0 &= c, \end{aligned} \tag{2.1}$$

where $c \in \mathbb{R}^d$. Let the system (2.1) describe the movement of a particle in space.

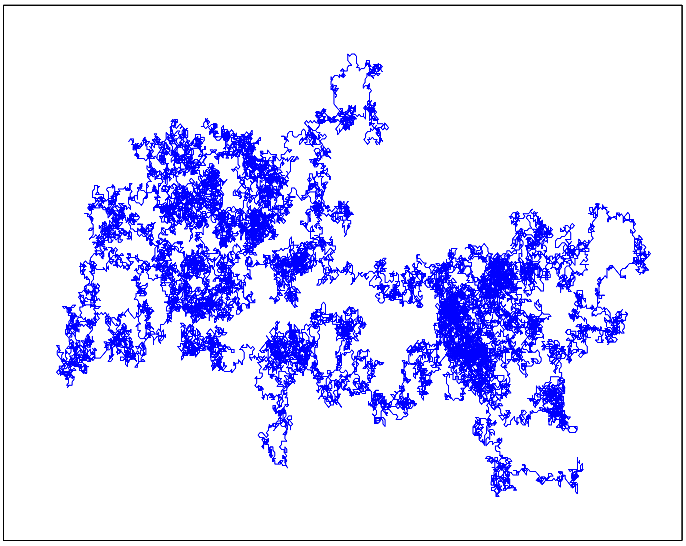

Figure 1: The erratic motion of a particle under nearest neighbor interactions produces a fractal like path. The curve traced out by the particle is called a random walk or Brownian motion.

Next, we would like to introduce this particle to an environment where other particles collide causing the particle to move in any direction with certain probabilities, as seen in figure (2). Mathematically, we would like to add "noise" to the system (2.1). That is,

$$\frac{dx_t}{dt} = f(t, x_t) + g(t, x_t)\frac{dW_t}{dt}, \tag{2.2}$$

where $x_0 = c$ and $\{W_t\}$ is called the Wiener process (or Brownian motion) on a probability space $(\Omega, \mathcal{F}, P)$. The last term in equation (2.2) does not make sense, since the Wiener process is not differentiable anywhere. We now try to make sense of this term.

3 Stochastic Differential Equations and the Itô Formula

Definition 3. *Define the Wiener process as a sequence of random variables $\{W_t\}$ on a probability space $(\Omega, \mathcal{F}, P)$ with the properties,*

1. *$W_0 = 0$, and the mapping $t \mapsto W_t$ is continuous with probability one.*
2. *For $t_0 < t_1 < ... < t_n$, the increments $W_{t_0}, W_{t_1} - W_{t_0}, ..., W_{t_n} - W_{t_{n-1}}$, are independent.*
3. *$W_t - W_s \sim N(0, t-s)$. That is $W_t - W_s$ has a normal distribution with mean zero and standard deviation $t - s$.*

The movement is described by the Wiener process. Named after mathematician Norbert Wiener who explained Brownian motion mathematically. Now we wish to derive the chain rule for the stochastic differential equation. To do so, consider the chain rule in deterministic systems. For example, let

$$\frac{dx_t}{dt} = f(t, x_t), \quad x_0 = c \in \mathbb{R}^d. \tag{3.1}$$

Consider a differentiable function $V(t, x_t)$. To find the derivative with respect to time we make the following calculations.

$$\begin{aligned} \frac{\partial}{\partial t}\left(V(t, x_t)\right) &= \frac{\partial V}{\partial t} + \sum_{j=0}^{d} \frac{\partial V}{\partial x_j} \frac{d(x_t)_j}{dt} \\ &= \frac{\partial V}{\partial t} + \sum_{j=1}^{d} \frac{\partial V}{\partial x_j} f_j(t, x_t) = LV, \end{aligned}$$

where f_j is the j^{th} compontent of f. You can use this to show existence and uniqueness of ODE's with out the condition of Lipschitz boundedness. Do derive the differential operator L in the random setting we need a new chain rule. Instead of writing the differential equation in (2.2), where dW_t/dt does not make sense, we instead write

$$dx_t = f(t, x_t)\ dt + g(t, x_t)\ dW_t, \tag{3.2}$$

or

$$x_t = x_0 + \int_0^t f(s, x_s)\ ds + \int_0^t g(s, x_s)\ dW_s,$$

where the stochastic integral is made precise by using random step functions. In the $\mathbb{R}^d$ setting, $W_t = (W_t^{(1)}, \ldots, W_t^{(d)})$ where each component is a Wiener process independent of all other components. Let $\phi \in C_1^2([0, \infty) \times \mathbb{R}^d)$ be a function mapping into the reals, and define $Y_t = \phi(t, x_t)$. Then the chain rule is defined using Itô's formula as

$$dY_t = \frac{\partial \phi(t, x_t)}{\partial t}\ dt + \sum_{i=1}^{d} \frac{\partial \phi(t, x_t)}{\partial x_i}\ d(x_t)_i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 \phi}{\partial x_i \partial x_j}\ d(x_t)_i d(x_t)_j. \tag{3.3}$$

To find the numerous terms of $d(x_t)_i d(x_t)_j$ we can use the following table,

x	dt	$dW_t^{(1)}$	$dW_t^(2)$	$\cdots$	$dW_t^{(d)}$
dt	0	0	0	$\cdots$	0
$dW_t^{(1)}$	0	dt	0	$\cdots$	0
$dW_t^{(2)}$	0	0	dt	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
$dW_t^{(d)}$	0	0	0	$\dots$	dt

4 Examples

Now we use the Itô formula to compute a couple of examples.

Example 1. *Let* $\alpha_t = \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right)$. *Then by the Itô formula in (3.3) and the table above we have*

$$\begin{aligned} d\alpha_t &= -\frac{\sigma^2}{2}\alpha_t\, dt + \sigma\alpha_t\, dW_t + \frac{\sigma^2}{2}\alpha_t\, dt \\ &= \sigma\alpha_t\, dW_t. \end{aligned}$$

Here α_t *can be used to model the price of a stock.*

Example 2. *Consider the following two dimensional system,*

$$\begin{aligned} dx_t &= x_t^3 y_t\, dt + \epsilon dW_t^{(1)}, \\ dy_t &= 3x_t y_t\, dt + \epsilon dW_t^{(2)}. \end{aligned}$$

This system describes the motion of a particle in two dimensions with smaller particles colliding with it causing the epsilon Brownian motion terms. Let $B_t = \phi(t, x_t, y_t)$. *Then by Itô's formula and the table above we have*

$$\begin{aligned} dB_t &= \frac{\partial\phi}{\partial t}\, dt + \frac{\partial\phi}{\partial x}\, dx_t + \frac{\partial\phi}{\partial y}\, dy_t + \frac{\epsilon^2}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\, dt \\ &= \left[\frac{\partial\phi}{\partial t} + \frac{\epsilon^2}{2}\left(\phi_{xx} + \phi_{yy}\right)\right]\, dt + \frac{\partial\phi}{\partial x}\left(x_t^3 y_t\, dt + \epsilon\, dW_t^{(1)}\right) + \frac{\partial\phi}{\partial y}\left(3x_t y_t\, dt + \epsilon dW_t^{(2)}\right) \\ &= \left[\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}x_t^3 y_t + \frac{\partial\phi}{\partial}3x_t y_t + \frac{\epsilon^2}{2}\Delta\phi\right]\, dt + \epsilon\frac{\partial\phi}{\partial x}\, dW_t^{(1)} + \epsilon\frac{\partial\phi}{\partial y}\, dW_t^{(2)}. \end{aligned}$$

We can write this differential in equation form to get

$$B_t = B_0 + \int_0^t L\phi\, dt + \int_0^t \epsilon\frac{\partial\phi}{\partial x}\, dW_s^{(1)} + \int_0^t \epsilon\frac{\partial\phi}{\partial y}\, dW_s^{(2)}. \tag{4.1}$$

If we look at the expected value of B_t, *the last two stochastic integrals in (4.1) are zero by martingale theory. Thus we have*

$$E[B_t] = B_0 + \int_0^t L\phi\, dt,$$

where the operator L *is elliptic on all of* $\mathbb{R}^2$.

5 More Elliptic Operators

Remark 1. *The differential operator*

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \quad \text{with } a_\alpha \in C^\infty(\mathbb{R}^d),$$

on $\mathbb{R}^d$ *is elliptic on* $\mathbb{R}^d$ *if the principle symbol,*

$$P_L = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \neq 0,$$

for $\xi \in \mathbb{R}^d$ *whenever* $|\xi| \neq 0$*. Also this definition can be extended to some domain* $U \subseteq \mathbb{R}^d$*.*

Example 3. *Consider Laplace's operator denoted* Δ*, where*

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

Then the principle symbol is

$$P_\Delta = \|\xi\|_2^2 \neq 0,$$

for all $\xi \neq 0$*. Thus any operator of the form* $L =$*lower order terms* $+\Delta$ *is elliptic.*

Example 4. *Consider the operator*

$$L = (x^2 - y_2 + 1)\frac{\partial}{\partial x} + \epsilon \frac{\partial^2}{\partial y^2} + 2xy \frac{\partial}{\partial y},$$

which is not elliptic everywhere. To see this, the principle symbol is $P_L = \epsilon \xi_2^2$*. Thus any* $\xi = (a, 0)$*,* $a > 0$*, has principle symbol zero. However,* $Lf = g \in C^\infty(\mathbb{R}^2)$ *implies* $f \in C^\infty(\mathbb{R}^2)$ *which is known as hyper ellipticity, which is weaker than ellipticity.*

Definition 4. *If* L *is a differential operator with the property that*

$$Lf = g \in C^\infty(U) \implies f \in C^\infty(U),$$

then L *is called hypoelliptic on* U*.*

Now we state a Theorem due to Hörmander in 1967.

Theorem 5.1. *Let* L *be a second order differential operator of the form*

$$L = X_0 + \sum_{j=1}^{\ell} X_j^2,$$

where the X_i*,* $i = 0, 1, ..., \ell$ *are first-order homogeneous differential operators with analytic coefficients. If the vector fields*

$$\begin{array}{rl} X_i, & i = 0, 1, ..., \ell \\ \{X_{i_1}, X_{i_2}\}, & i_1, i_2 = 0, ..., \ell \\ \{X_{i_1}, \{X_{i_2}, X_{i_3}\}\}, & i_1, i_2, i_3 = 0, ..., \ell \\ \vdots & \vdots \end{array}$$

span the tangent space on U*. Then* L *is hypoelliptic on* U*.*

The brackets $\{\cdot, \cdot\}$, are Lie brackets.

Example 5. *Let*

$$L = (x^2 - y^2 + 1)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + \epsilon\frac{\partial^2}{\partial y^2}.$$

Here the vector fields in the theorem are

$$X_0 = (x^2 - y^2 + 1)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$$
$$X_1 = \sqrt{\epsilon}\frac{\partial}{\partial y}.$$

Notice that the vector fields X_0, and X_1 do not span the tangent space. Thus we must compute Lie brackets.

$$\{X_0, X_1\} = X_0X_1 - X_1X_0 = 2\sqrt{\epsilon}y\frac{\partial}{\partial x} - 2\sqrt{\epsilon}x\frac{\partial}{\partial y}$$
$$\{X_1, \{X_0, X_1\}\} = X_1\{X_0, X_1\} = 2\epsilon\frac{\partial}{\partial x}.$$

Therefore X_1 and $\{X_1, \{X_0, X_1\}\}$ span the tangent space. This example is called the infinitesimal generator of

$$\begin{cases} dx_t &= (x_t^2 - y_t^2 + 1)\ dt \\ dy_t &= 2x_ty_t\ dt + \sqrt{\epsilon}\ dW_t \end{cases}$$

If we define $p_t(x_t, y_t)$ as the joint density of the random variables x_t and y_t, that is

$$P[x_t \in A, y_t \in B] = \int_A\int_B p_t(x_t, y_t)\ dx\ dy,$$

then

$$\frac{\partial p_t}{\partial t} = L^*p_t.$$

Therefore

$$\frac{\partial p_t}{\partial t} - L^*p_t = 0,$$

satisfies the above theorem.