Research Statement

Ji Li[*,†]

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Introduction

My research is in the area of enumerative and algebraic combinatorics. More specifically, I focus on enumerating graphs using the combinatorial theory of species as a framework.

The combinatorial theory of species was initiated by Joyal in [5]. A *species* is a functor from the category of finite sets with bijections to itself (see [2]). A species F generates for each finite set U a finite set $F[U]$, called the set of F-*structures* on U, and for any bijection σ from U to V a bijection $F[\sigma]$ from $F[U]$ to $F[V]$. (See Figure 1 for an example of transport of structures.)

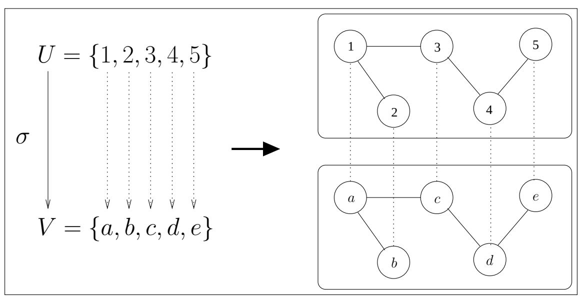

Figure 1: $\sigma : U \to V$ induces a bijection $\mathcal{G}[\sigma]$ sending each graph on U to a graph on V.

We denote by $\mathfrak{S}_n$ the *symmetric group* of order n, i.e., the set of permutations on $[n] = \{1, 2, \dots, n\}$. The group $\mathfrak{S}_n$ acts on the set $F[n] = F[\{1, 2, \dots, n\}]$ by *transport of structures*. The $\mathfrak{S}_n$-orbits under this action are called *unlabeled* F-structures of order n.

We apply operations on species to generate new species, and the operations of species translate into operations of the generating series of species systematically (see [2, pp. 1–58] for details). The species operations that are frequently used in this report are the *sum* $\Phi + \Psi$, the *product* $\Phi\Psi$ or $\Phi \cdot \Psi$, and the *composition* $\Phi(\Psi)$ or $\Phi \circ \Psi$ of species Φ and Ψ.

We consider only simple graphs (without loops or multiple edges). We denote by $\mathcal{G}$ the species of graphs. An *unlabeled graph* is formally defined as an isomorphism class of graphs. We denote by $\mathcal{E}$ the species of *sets*, and this is the same as the species of *edgeless graphs*. We denote by $\mathcal{K}$ the species of *complete graphs*.

[*] *Email address*: jli@math.arizona.edu
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I Enumeration of Bi-Point-Determining Graphs

I.1 Superimposition of Graphs

Definition I.1. *([4]) Let $H_1, \ldots, H_m$ be graphs with disjoint vertex sets, and let G be a graph with vertex set $\{V(H_1), \ldots, V(H_m)\}$. We define the superimposition $G|_{H_1,\ldots,H_m}$ of G on $\{H_1, \ldots, H_m\}$ to be the graph with vertex set $\bigcup_{i=1}^m V(H_i)$ in which $\{u, v\}$ is an edge if it is an edge of some H_i or if $u \in V(H_i)$ and $v \in V(H_j)$ for some $i \neq j$, and $\{V(H_i), V(H_j)\} \in E(G)$.*

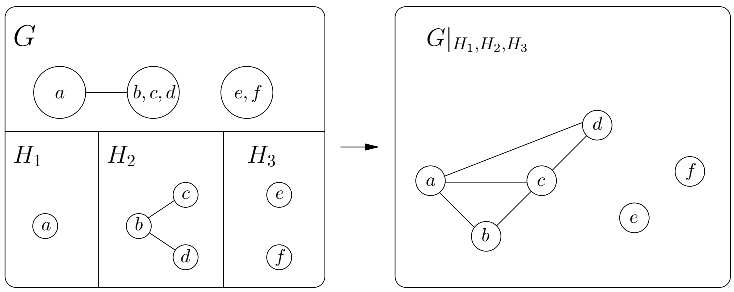

Figure 2: The superimposition $G|_{H_1,H_2,H_3}$.

Let Φ and Ψ be two species of graphs; i.e., for every finite set U, $\Phi[U]$ and $\Psi[U]$ are sets of graphs with vertex set U. We define a species $\Phi \diamond \Psi$ for which $(\Phi \diamond \Psi)[U]$ is the set of all superimpositions $G|_{H_1,\ldots,H_m}$ in which $H_1, \ldots, H_m$ are Ψ-graphs with $\bigcup_{i=1}^m V(H_i) = U$ and G is a Φ-graph with vertex set $\{V(H_1), \ldots, V(H_n)\}$.

Proposition I.1. *([4]) Let Φ and Ψ be species such that every $\Phi \diamond \Psi$-graph can be expressed uniquely as a superimposition of a Φ-graph on a set of Ψ-graphs. Then $\Phi \circ \Psi$ is isomorphic to $\Phi \diamond \Psi$.* □

I.2 Point-determining graphs

The *point-determining graphs*, previously studied by Sumner [11] and Read [9], are graphs in which any two distinct vertices have distinct *neighborhoods*. If we start with any graph, and identify vertices with the same neighborhood, we obtain a point-determining graph. Point-determining graphs (both labeled and unlabeled) were counted by Read [9].

We obtain Theorem I.2 by showing that any graph G can be expressed uniquely as a superimposition of a point-determining graph on a set of edgeless graphs.

Theorem I.2. *(*Read*) For species $\mathcal{G}$ of graphs, species $\mathcal{P}$ of point-determining graphs, species $\mathcal{E}_+$ of nonempty edgeless graphs, we have*

$$\mathcal{G} = \mathcal{P} \circ \mathcal{E}_+. \tag{I.2}$$

I.3 Bi-point-determining graphs

Complements of point-determining graphs, which we call*co-point-determining graphs*, are graphs in which no two vertices have the same closed neighborhood. We denote by $\mathcal{B}$ the species of *bi-point-determining graphs*, which are graphs that are both point-determining and co-point-determining. These graphs may also be characterized by the property that their automorphism groups contain no transpositions; i.e., they are not fixed by switching any pair of vertices.

Just as an arbitrary graph can be reduced to point-determining graph by identifying graphs with the same neighborhood, an arbitrary graph may be reduced to a bi-point-determining graph by a more complicated compression in which the fibers are *cographs*, which are graphs obtained from edgeless graphs by complementation and union (previously studied by Corneil, Lerchs and Burlingham in [3]). Theorem I.3 is proved by showing that every graph can be expressed uniquely as a superimposition of a bi-point-determining graph on a set of cographs.

Theorem I.3. *([4]) The species $\mathcal{G}$ of graphs is the composition of the species $\mathcal{B}$ of bi-point-determining graphs and $\mathcal{C}$ of cographs. That is,*

$$\mathcal{G} = \mathcal{B} \circ \mathcal{C}. \tag{I.3}$$

II Enumeration of Prime Graphs

II.1 Cartesian Product of Graphs

The *(Cartesian) product* of two graphs G_1 and G_2, denoted $G_1 \odot G_2$, is the graph whose vertex set is the Cartesian product of the vertex set of G_1 and the vertex set of G_2, in which (u, v) is adjacent to the vertex (w, z) if either $u = w$ and $\{v, z\} \in E(G_2)$ or $v = z$ and $\{u, w\} \in E(G_1)$. A connected graph G is called *prime* with respect to Cartesian multiplication if G is not a single vertex and G cannot be factored non-trivially. Sabidussi [10] showed that any non-trivial connected graph can be uniquely decomposed into prime factors up to isomorphism. (See Figure 3 for an example.)

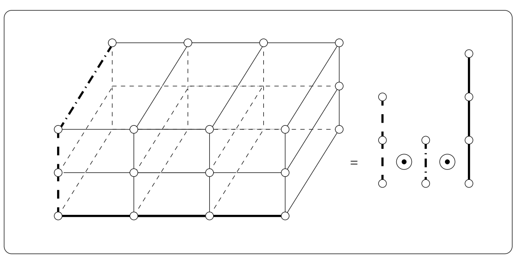

Figure 3: The decomposition of a connected graph into its prime factors.

II.2 Enumeration on a Free Commutative Monoid

The unique prime decomposition of connected graphs gives a *free commutative monoid* structure on the set of unlabeled connected graphs, denoted $\mathbb{M}$, generated by the set of unlabeled prime graphs $\mathbb{P}$. As a consequence of Sabidussi's theorem we can count prime graphs using Dirichlet series, which are series of the form $\sum_{i\in I} a_i/i^s$:

$$\sum_{G\in\mathbb{M}} \frac{1}{l(G)^s} = \prod_{P\in\mathbb{P}} \frac{1}{1-l(P)^{-s}},$$

where for any graph G, $l(G)$ denotes the number of vertices in G. This infinite product is analogous to the infinite product of the zeta function, since the set of natural numbers $\mathbf{N}$ has a free commutative monoid structure with a generating set $\mathbf{P}$, the set of prime numbers:

$$\zeta(s) = \sum_{n\in\mathbf{N}} \frac{1}{n^s} = \prod_{p\in\mathbf{P}} \frac{1}{1-p^{-s}}.$$

Extending the work of Raphaël Bellec [1], we get a formula for the number b_n of unlabeled prime graphs on n vertices:

Theorem II.1. *([6]) Let c_n be the number of unlabeled connected graphs on n vertices, and let d_n be the numbers such that $\sum_{n\geq 1} d_n/n^s = \log \sum_{n\geq 1} c_n/n^s$. Then we have*

$$b_n = \frac{1}{e}\sum_{l|e} \mu\left(\frac{e}{l}\right) l d_{r^l},$$

where e is the largest number such that $n = r^e$ for some r.

II.3 Exponential composition of species

For F a species with $F[\emptyset] = \emptyset$, we write $F^{\boxdot k}$ for the arithmetic product of k copies of F. An $F^{\boxdot k}$-structure on U is a tuple of the form $((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k))$, where $(\pi_1, \pi_2, \ldots, \pi_k)$ is a k-*rectangle* on U, and f_i is an F-structure on π_i (see [7] by Maia and Méndez for detailed definitions of the k-rectangles and the arithmetic product of species). The symmetric group $\mathfrak{S}_k$ of order k acts on the subscripts of π_i and f_i *naturally*, and we get a *quotient species* [2, p. 159] under this group action.

Let A be a permutation group of order m. That is, A is a subgroup of $\mathfrak{S}_n$. Then A acts on the set of $(X^n/B)^{\boxdot m}$-structures, where X^n/B is the *molecular species* corresponding to the subgroup B of $\mathfrak{S}_n$ (see [12] by Yeh). Figure 4 illustrates this group action.

Definition II.1. *([6]) The* exponential composition *of F of* order k*, denoted $\mathcal{E}_k\langle F\rangle$, is the quotient species $F^{\boxdot k}/\mathfrak{S}_k$. For species F with $F[\emptyset] = F[1] =$, we define the*exponential composition *of F, denoted $\mathcal{E}\langle F\rangle$, to be the sum of $\mathcal{E}_k\langle F\rangle$ on all nonnegative integers k, i.e.,*

$$\mathcal{E}\langle F\rangle = \sum_{k\geq 0} \mathcal{E}_k\langle F\rangle.$$

Theorem II.2. *([6]) Let $\mathcal{C}$ and $\mathcal{P}^r$ be species of connected graphs and species of prime graphs, respectively. We have*

$$\mathcal{C} = \mathcal{E}\langle \mathcal{P}^r\rangle.$$

We generalize a theorem by Palmer and Robinson [8, p. 128] to get a formula for the cycle index of the exponential composition of a molecular species of order k, which, together with the above theorem, enables us to enumerate the species of prime graphs $\mathcal{P}^r$.

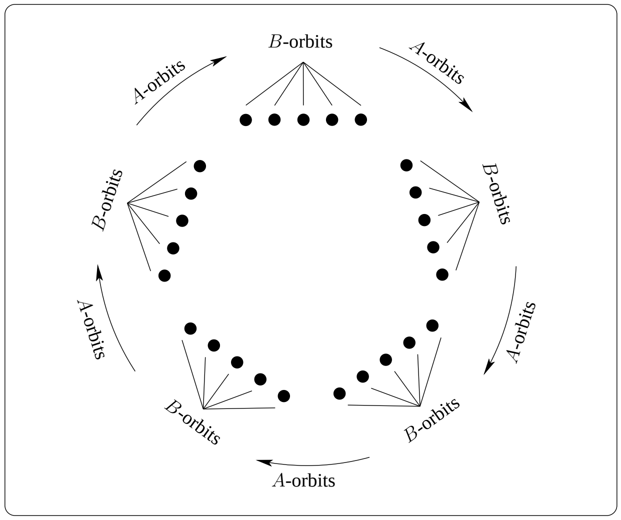

Figure 4: $((X^n/B)^{\boxdot m})/A = X^{n^m}/B^A$.

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