Counting Bi-Point-Determining Graphs

A Bijection

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Outline

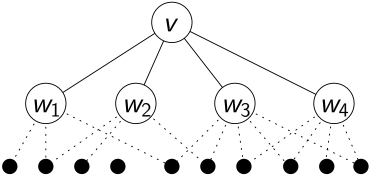
1. Point-Determining Graphs
2. Bi-Point-Determining Graphs
3. A Bijection for Bi-Point-Determining Graphs
4. Generating Functions

Neighborhood of a Vertex

Definition

In a graph G, the *neighborhood* of a vertex v is the set of vertices adjacent to v, the *augmented neighborhood* of a vertex is the union of the vertex itself and its neighborhood.

Example



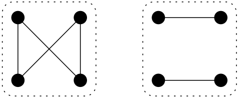
In the above figure, the neighborhood of vertex v is the set $\{w_1, w_2, w_3, w_4\}$, while the augmented neighborhood of v is the set $\{v, w_1, w_2, w_3, w_4\}$.

Point-Determining Graphs and Co-Point-Determining Graphs

Definition

- A graph is called *point-determining* if no two vertices of this graph have the same neighborhoods.
- A graph is called *co-point-determining* if its complement is point-determining.
- Equivalently, a graph is *co-point-determining* if no two vertices of this graph have the same *augmented neighborhoods*.

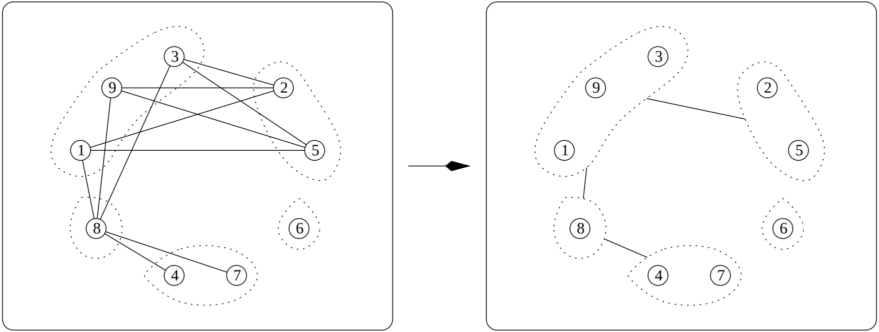
Example



The graph on the left is co-point-determining, and the graph on the right is point-determining. These two graphs are complements of each other.

A Transformation

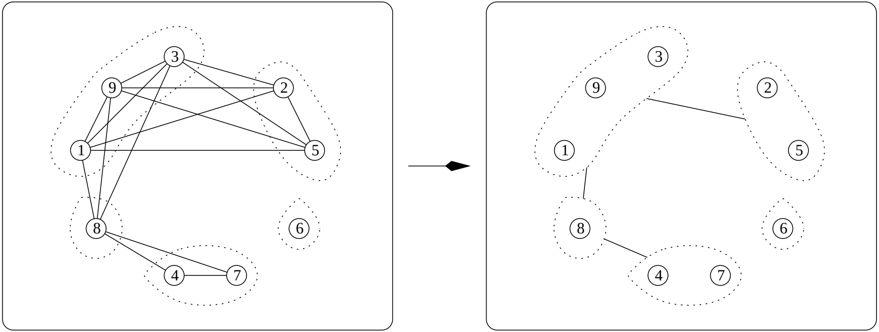
A transformation from a graph G to a point-determining graph P.



The above figure illustrates the transformation from a graph G with vertex set $[11]$ to a point-determining graph P with vertex set $\{\{1,9,3\},\{8\},\{4,7\},\{6\},\{2,5\}\}$.

Another Transformation

A transformation from a graph G to a co-point-determining graph Q.



Here is another similar transformation from a graph G with vertex set $[11]$ to a co-point-determining graph Q with vertex set $\{\{1,9,3\},\{8\},\{4,7\},\{6\},\{2,5\}\}$.

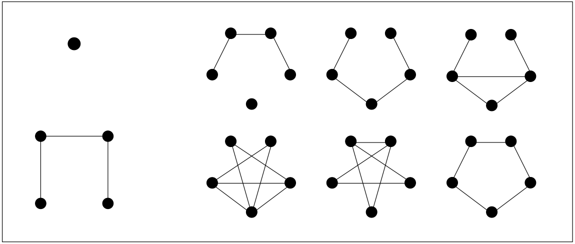
Outline

Bi-Point-Determining Graphs

Definition

A *bi-point-determining graph* is a graph that is both point-determining and co-point-determining.

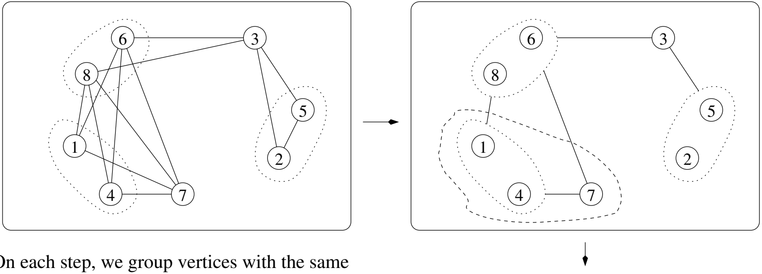
Example

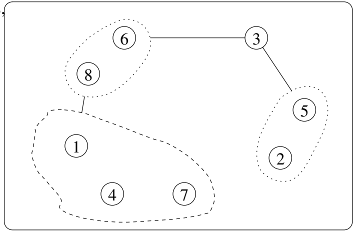


Listed in above are all unlabeled bi-point-determining graphs with no more than 5 vertices.

Transform a Graph into a Bi-Point-Determining Graph?

Yes.





On each step, we group vertices with the same neighborhoods or the same augmented neighborhoods, and get a new graph whose vertices are sets of vertices of the previous graph.

In this way, we end up with a graph in which no two vertices have the same neighborhood or the same augmented neighborhood, i.e., a bi–point–determining graph.

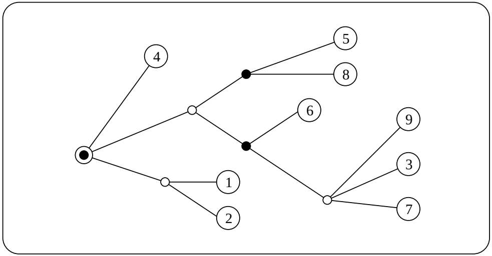
But, we would like to keep track of the procedure so that the original graph can be reconstructed....

Alternating Phylogenetic Trees

Definition

- A *phylogenetic tree* is a rooted tree with labeled leaves and unlabeled internal vertices in which no vertex has exactly one child.
- An *alternating phylogenetic tree* is either a single vertex, or a phylogenetic tree with more than one labeled vertex whose internal vertices are colored black or white, where no two adjacent vertices are colored the same way.

Example



An alternating phylogenetic tree on 9 vertices, where the root is colored black.

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An Informal Description

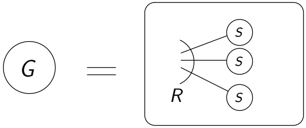
The claim is —

The structure of alternating phylogenetic trees can be used to keep track of the transformation of an arbitrary graph into a bi-point-determining graph.

We let

- G denote arbitrary graphs
- R denote arbitrary bi-point-determining graphs
- S denote alternating phylogenetic trees.

An illustration

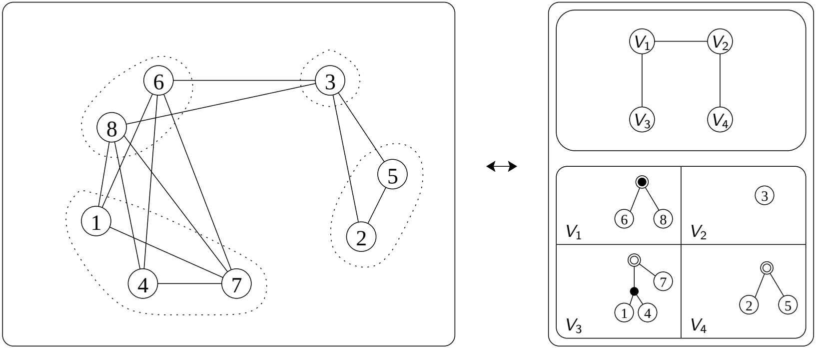


The above figure means..

the structure of graphs is the structure of bi-point-determining graphs superimposed with the structures of alternating phylogenetic trees.

An Illustration of the Bijection

Example



The Formal Description of the Bijection

For any finite set U, we construct a bijection between

the set of

graphs with vertex set U

and

the set of

triples of the form (π, R, γ) such that

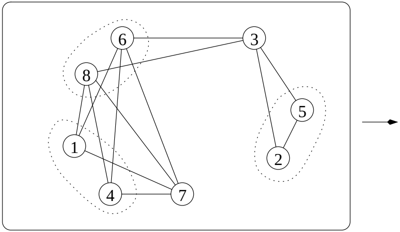
- π is a partition of the set U, i.e., $\pi = \{V_1, V_2, \ldots, V_k\}$
- R is a bi-point-determining graph with vertex set π
- γ is a set of alternating phylogenetic trees $\{S_1, S_2, \ldots, S_k\}$, where each S_i is an alternating phylogenetic tree with vertex set V_i.

Next, we will see..

- how to get a triple from an arbitrary graph
- how to construct a graph from a given triple

From a Graph to a Triple

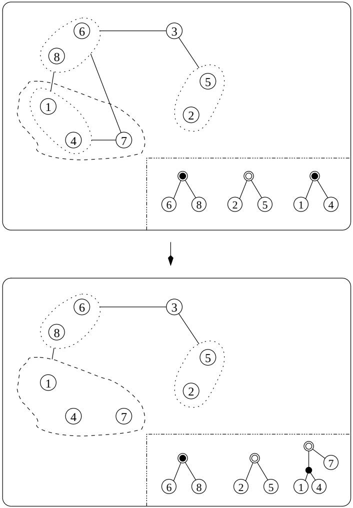
Example



On each step, we group vertices with the same neighborhoods or vertices with the same augmented neighborhods.

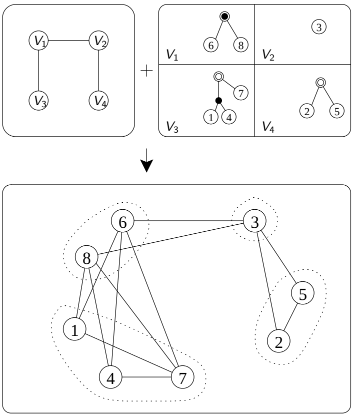
Whenever vertices with the same neighborhods are grouped, we connect the corresponding vertices/alternating phylogenetic trees with a black node.

Whenever vertices with the same augmented neighborhoods are grouped, we connected the corresponding vertices/ alternating phylogenetic trees with a white node.



From a Triple to a Graph

Example

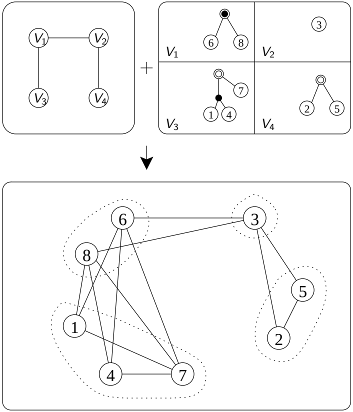


Given a triple (π, R, γ)*, where*

- $\pi = \{V_1, V_2, \ldots\}$ is a partition of U
- R is a bi-point-determining graph on the blocks of π
- γ is a set $\{S_1, S_2, \ldots\}$ in which each S_i is an alternating phylogenetic tree labeled on the set V_i.

From a Triple to a Graph: Continue

Example



Then there is a unique graph G with vertex set U such that...

Vertices v_1 and v_2 of G are adjacent **if and only if** exactly one of the following two conditions is satisfied:

a) v_1 and v_2 are labels of vertices of S_i for some i, and the common ancestor of v_1 and v_2 in S_i is colored white.

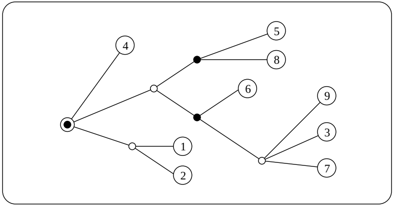
b) $v_1 \in V_i$, $v_2 \in V_j$, and V_i and V_j are adjacent vertices in the bi-point-determining graph R.

The Common Ancestor

Definition

The *common ancestor* of two vertices a and b in a phylogenetic tree is defined to be such that if we take the unique shortest path from a to b, say, $w_0 w_1 \cdots w_l$, with $w_0 = a$ and $w_l = b$, then the common ancestor of a and b is the unique w_i for which both w_{i-1} and w_{i+1} are children of w_i.

Example

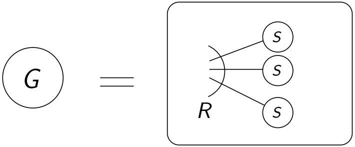


The common ancestor of vertices 5 and 4 is colored black, while the common ancestor of vertices 5 and 3 is colored white.

Outline

This Bijection Gives Rise to Functional Equations

As illustrated..



We get

$$G = R \circ S.$$

Which reads..

The structure of graphs is the structure of bi-point-determining graphs composed with the structure of alternating phylogenetic trees.

The Exponential Generating Function for Labeled Bi-Point-Determining Graphs

We write

- $R(x)$ to be the exponential generating function of labeled bi-point-determining graphs;
- $G(x)$ to be the exponential generating function of labeled graphs, which is given by

$$G(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}.$$

Then we get

$$R(x) = G(2\log(1+x) - x).$$

$$R(x) = \frac{x}{1!} + 12\frac{x^4}{4!} + 312\frac{x^5}{5!} + 13824\frac{x^6}{6!} + 1147488\frac{x^7}{7!} + 178672128\frac{x^8}{8!} + \cdots.$$

The Ordinary Generating Function for Unlabeled Bi-Point-Determining Graphs

We write

$\widetilde{R}(x)$ to be the (ordinary) generating function of unlabeled bi-point-determining graphs. Then

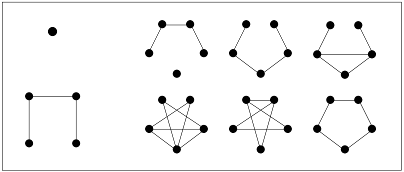
$$\widetilde{R}(x) = Z_G(x - 2x^2, x^2 - 2x^4, \dots).$$

Here Z_G is the so-called *cycle index* of the structures of graphs G, and the formula for Z_G is known.

We write down the beginning terms of $\widetilde{R}(x)$

$$\widetilde{R}(x) = x + x^4 + 6x^5 + 36x^6 + 324x^7 + 5280x^8 + \cdots .$$

Compare with..



The End

Thank you for your patience!