On Uniqueness of Limit Cycle for the Equation: $\ddot{x} + f(x)\phi(\dot{x})\dot{x} + g(x) = 0$

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In this paper, we use the theory of generalized rotated fields to prove a theorem of the uniqueness of limit cycle for the generalized Liénard system. The damping terms of this system are dependent on $\dot{x}$.

Our result extends a famous Zhang-Zhifen's theorem on the uniqueness problem of limit cycle, which is contributed the Liénard systems whose damping terms are independent of $\dot{x}$.

Keywords: Liénard equation, Limit cycle.

1. INTRODUCTION

The general autonomous equation of Liénard type

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \tag{1.1}$$

was first studied by Levinson and Smith in the classical paper [1], and later many researches have contributed to the theory of this equation with respect to existence and uniqueness of nontrivial periodic solutions. The books by Sansone and Conti [2] and Z. F. Zhang et al. [3] contain an excellent summary of the extensive results on such problems. On the uniqueness problem of limit cycle for planar systems, almost all existing results are on Liénard systems. Moreover, most uniqueness results are contributed the systems whose damping terms are independent of $\dot{x}$, that is,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1.2}$$

The reader can refer to [4, 5, 6, 7] or [3, Chapter 4, Sect. 4]. Only a few papers have studied the uniqueness problem of limit cycle for Liénard systems with damping terms dependent on $\dot{x}$ (see [8, 9, 10]). Among all uniqueness theorems, Zhang's theorem has particular significance because it has a simple easy-checking condition and extensive application. We state it here.

ZHANG'S THEOREM. *Let $f(x)$ and $g(x)$ be continuous differentiable on R. Suppose that the system* (1.2) *satisfies the following conditions:*

(C_1) $xg(x) > 0,\ x \neq 0,$ *and* $G(\pm\infty) = \pm\infty$ *with* $G(x) = \int_0^x g(\sigma)d\sigma$,

(C_2) $\frac{f(x)}{g(x)}$ *is increasing for* $x \in (-\infty, 0), (0, +\infty),$ *and* $\frac{f(x)}{g(x)} \neq 0$ *in a neighborhood of the origin.*

Then (1.2) *has at most one limit cycle. Moreover, the limit cycle is stable if it exists.*

The purpose of the present paper is to study the uniqueness problem of limit cycle for the following Liénard system

$$\ddot{x} + f(x)\phi(\dot{x})\dot{x} + g(x) = 0, \tag{1.3}$$

whose damping term is dependent on $\dot{x}$. Using the theory of rotated vector fields and Tkachev's result in [11], we shall verify a Zhang's type uniqueness theorem, which is stated as follows:

THEOREM A. *Let $f(x), g(x)$ and $\phi(y)$ be continuously differentiable on R. Assume that* (1.3) *satisfies the following conditions:*

(1)$xg(x) > 0,\ x \neq 0,$ *and* $G(\pm\infty) = \pm\infty$ *with* $G(x) = \int_0^x g(\sigma)d\sigma$,

(2)$\frac{d}{dx}\frac{f(x)}{g(x)} \geq 0,\ x \in (-\infty, 0), (0, +\infty),$ *and* $\frac{f(x)}{g(x)} \neq 0$ *in a neighborhood of the origin,*

(3)$\frac{d}{dy}(y\phi(y)) \geq 0,\ y \in (-\infty, +\infty)$ *and* $\phi(y) \neq 0$ *in a neighborhood of the origin.*

Then (1.3) *has at most one limit cycle, and if it exists, it must be stable.*

Obviously, when $\phi(y) \equiv 1$, Theorem A is Zhang's result.

2. THE PROOF OF THEOREM A

Before giving the proof of Theorem A, we introduce the concept of generalized rotated vector fields and Tkachev's result.

DEFINITION 1. *The vector fields $(P(x, y, a), Q(x, y, a))$ is called a family of generalized rotated vector fields, where $a \in I \subset R$, if*

$$[Q(x, y, a)P(x, y, a_0) - P(x, y, a)Q(x, y, a_0)]\ \mathrm{sgn}(a - a_0) \geq 0 (\leq 0) \tag{2.1}$$

for all $x, y \in R$ and $a, a_0 \in I$. (see[1], Chapter 4, Sect. 3).

THEOREM B. (Tkachev) *Consider the planar autonomous system*

$$\begin{cases} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y), \end{cases} \tag{2.2}$$

where P(x, y) and Q(x, y) are defined in a simply connected region $G \subseteq R^2$ and are of class C^2. Suppose that P and Q satisfy the following conditions:

(1) $P(x,y) = 0$ and $Q(x,y) = 0$ can be represented by the monotone functions of types $y = f(x)$ and $x = g(y)$, respectively;

(2) Let

$$M(x,y) = \frac{1}{Q}\frac{\partial Q}{\partial y}, \quad A(x,y) = \frac{\partial M}{\partial x}, \qquad \text{for } Q \neq 0, \tag{2.3}$$

$$N(x,y) = \frac{1}{P}\frac{\partial P}{\partial x}, \quad B(x,y) = -\frac{\partial N}{\partial y}, \qquad \text{for } P \neq 0. \tag{2.4}$$

Then the functions $A(x,y)$ and $B(x,y)$ has the same sign in G and never change sign. Moreover, in any subregion of G, either A or B is not identically zero. It is also assumed that near the upper (lower) half of $y = f(x)$, $B(x,y)N(x,y) \leq 0 (\geq 0)$, and that near the right (left) side of $x = g(y)$, $A(x,y)M(x,y) \leq 0 (\geq 0)$. Then the system (2.2) *has no multiple limit cycle in G, where no multiple limit cycle means a limit cycle whose characteristic exponent is nonzero.* (see[11], Theorem 3, Case 1).

Now, we begin with the proof of Theorem A.

First, we claim that if Theorem A is valid for the system with $g(x) = x$, then the general theorem is still true. In order to prove this, let us make transformation. It is easy to see that the system (1.3) is equivalent to the planar system

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)\phi(y)y. \end{cases} \tag{2.5}$$

Let $u = \sqrt{2G(x)} \cdot \text{sgn}(x)$, where $G(x) = \int_0^x g(\sigma)d\sigma$. Then

$$\begin{aligned} \frac{du}{dx} &= \frac{g(x)}{u}, \\ \frac{du}{dy} &= \frac{y}{-u - \frac{f(x)}{g(x)}u\phi(y)y}, \end{aligned}$$

where $x = x(u)$ is the inverse function of $u = \sqrt{2G(x)} \cdot \text{sgn}(x)$. This shows that (2.5) is equivalent to

$$\begin{cases} \dot{u} &= y \\ \dot{y} &= -u - f_1(u)\phi(y)y, \end{cases} \tag{2.6}$$

where $x = x(u)$, $f_1(u) = \frac{f(x(u))}{g(x(u))}u$. Because

$$\begin{aligned} \frac{d}{du}\frac{f_1(u)}{u} &= \frac{d}{du}\frac{f(x(u))}{g(x(u))} \\ &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\frac{dx}{du} \\ &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\frac{u \cdot x}{g(x) \cdot x} \\ &\geq 0, \end{aligned}$$

It is easy to know that the general Theorem A holds if it is true in the special system (2.6).

Proof of Theorem A. From the previous description, without loss of generality, we can assume that $g(x) = x$.

At first, we construct the system

$$\ddot{x} + f(x,a)\phi(\dot{x})\dot{x} + x = 0, \tag{2.7}$$

which is equivalent to

$$\begin{cases} \dot{x} = y = P(x,y,a) \\ \dot{y} = -x - f(x,a)\phi(y)y = Q(x,y,a), \end{cases} \tag{l_a}$$

where $f(x,a) = f(x) - a$, $a \geq 0$. Obviously, if $a = 0$, (l_a) is (2.5). Since

$$\frac{d}{dx}\frac{f(x,a)}{x} = \frac{d}{dx}\frac{f(x)}{x} + \frac{a}{x^2} \geq 0, \tag{2.8}$$

(l_a) satisfies the conditions (1) through (3) of Theorem A. Now we shall prove that (l_a) has at most one limit cycle. The proof is divided into three steps.

Step 1. Prove that (l_a) forms a family of generalized rotated vector fields with respect to the parameter $a \in R$.

At first, we shall prove that $\phi(y) \geq 0$ for all $y \in R$.

From the condition (3), we can easily know that

$$\phi(0) \geq 0 \quad \text{and} \quad y_1\phi(y_1) \geq y_2\phi(y_2), \quad y_1 > y_2, \quad \forall y_1, y_2 \in R. \tag{2.9}$$

Now we assume that there exists $y_0 \in R \setminus \{0\}$ such that $\phi(y_0) < 0$. If $y_0 > 0$, then $\phi(y_0)y_0 < 0 = \phi(y)y|_{y=0}$, contradicting (2.9). If $y_0 < 0$, then $\phi(y_0)y_0 > 0 = \phi(y)y|_{y=0}$, contradicting (2.9). So we get that $\phi(y) \geq 0$ for all $y \in R$.

By calculation, it follows from $\phi(y) \geq 0$ that

$$\begin{aligned}&[Q(x,y,a)P(x,y,a_0) - P(x,y,a)Q(x,y,a_0)]\ \mathrm{sgn}(a-a_0)\\ =\ &\phi(y)y^2\mathrm{sgn}(a-a_0)^2 \geq 0.\end{aligned}$$

This shows (l_a) forms a family of generalized rotated vector fields with respect to the parameters $a \in R$.

Step 2. Prove that (l_a) has no multiple limit cycle.

For system (l_a), by computing the corresponding functions in Theorem B, we obtain that

$$N(x,y) \equiv B(x,y) \equiv 0,$$

$$M(x,y) = \frac{1}{Q}\frac{\partial Q}{\partial y} = \frac{d(y\phi(y))}{dy}\frac{f(x,a)}{x + f(x,a)\phi(y)y},$$

$$A(x,y) = \frac{\partial M}{\partial x} = \frac{d(y\phi(y))}{dy}\frac{d}{dx}\left(\frac{f(x,a)}{x}\right)\left(\frac{x}{x + f(x,a)\phi(y)y}\right)^2 \geq 0.$$

Suppose $f(x) \leq 0$ for all $x \in R$. Then we define a Liapunov function

$$V(x,y) = x^2 + y^2.$$

Along any solution of (l_a), we get that

$$\left.\frac{dV}{dt}\right|_{(l_a)} = -f(x)\phi(y)y^2 \geq 0.$$

It follows from Liapunov theorem that any solution of (l_a) tends to the origin as $t \to -\infty$. Therefore, in this case, there are no closed orbits.

In the following, we assume that $f(R) \cap (0,+\infty) \neq \emptyset$, and let $I = f(R) \cap [0,+\infty)$. It follows from Step 1 that (l_a) with $a \in I$ forms a family of generalized rotated vector fields. For any $a \in I$, there exists $b \in R$ with $b \neq 0$ such that $f(b) = a$. From (2.8) it follows that $f(x) - a > 0$ for all x with $|x| > |b|$. Therefore, there exist at most two nonzero solutions for the equation

$$f(x) - a = 0. \tag{2.10}$$

1°. There are two solutions $b < 0 < c$ for the equation (2.10). Then $Q(x,y,a) = 0$ consists of the following three branches:

$$L_{11} : x = x_{11}(y),\ 0 < y < +\infty;$$

$$L_{12} : x = x_{12}(y),\ \ -\infty < y < 0;$$

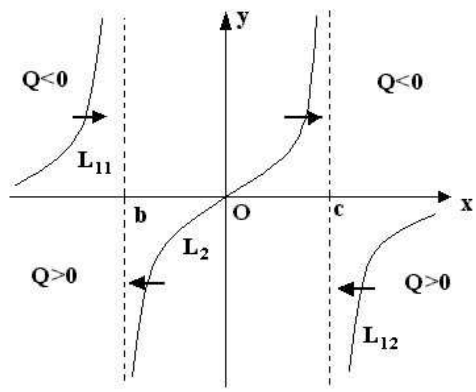

FIGURE 1

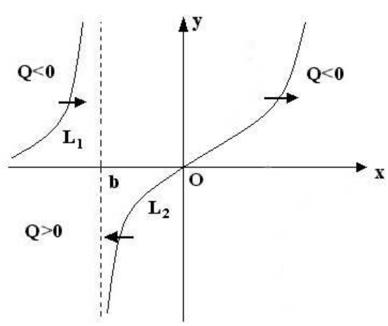

FIGURE 2

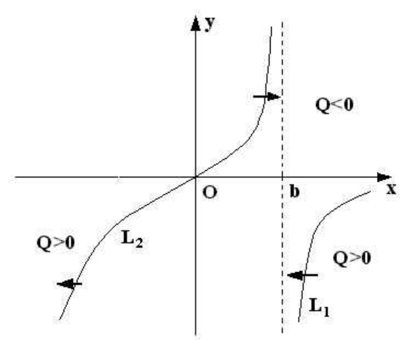

FIGURE 3

$$L_2 : x = x_2(y), \quad -\infty < y < +\infty.$$

L_{11}, L_{12}, L_2 and the direction of the vector field for (l_a) are sketched in Fig. 1.

Set $G = G_1 \cup G_2 \cup X$, where

$$G_1 = \{(x, y) : y > 0, x_{11}(y) < x < +\infty\},$$

$$G_2 = \{(x, y) : y < 0, -\infty < x < x_{12}(y)\},$$

$$X = \{(x, y) : y = 0, -\infty < x < +\infty\}.$$

Then from the direction of the vector field for (l_a) as shown in Fig. 1, and since the origin is the unique finite singular point of (l_a), it follows that if (l_a) has a limit cycle Γ, then Γ is contained in the simply connected domain G.

It is easy to check that in region $G_+ = \{(x, y) : x_2(y) < x < c, |y| < +\infty\}$, $M(x, y, a) \le 0$, and in region $G_- = \{(x, y) : b < x < x_2(y), |y| < +\infty\}$, $M(x, y, a) \ge 0$. Thus, $A(x, y, a)M(x, y, a) \le 0$ for $(x, y) \in G_+$ and $A(x, y, a)M(x, y, a) \ge 0$ for $(x, y) \in G_-$.

2°. There are only one solution $b \neq 0$ for the equation (2.10). If $b < 0$, then $Q(x, y, a) = 0$ consists of the following two branches:

$$L_1 : x = x_1(y),\ 0 < y < +\infty,$$

$$L_2 : x = x_2(y),\ \ -\infty < y < +\infty.$$

L_1, L_2 and the direction of the vector field for (l_a) are sketched in Fig. 2.

Set $G = G_1 \cup G_2$, where

$$G_1 = \{(x, y) : y > 0, x_1(y) < x < +\infty\},$$

$$G_2 = \{(x, y) : y \leq 0, -\infty < x < +\infty\}.$$

Then with the same method as 1°, we got that in region $G_+ = \{(x, y) : x_2(y) < x < +\infty, |y| < +\infty\}$, $M(x, y, a) \leq 0$, and in region $G_- = \{(x, y) : b < x < x_2(y), |y| < +\infty\}$, $M(x, y, a) \geq 0$. Thus, $A(x, y, a)M(x, y, a) \leq 0$ for $(x, y) \in G_+$ and $A(x, y, a)M(x, y, a) \geq 0$ for $(x, y) \in G_-$.

If $b > 0$, then $Q(x, y, a) = 0$ consists of the following two branches:

$$L_1 : x = x_1(y),\ \ -\infty < y < 0,$$

$$L_2 : x = x_2(y),\ \ -\infty < y < +\infty.$$

L_1, L_2 and the direction of the vector field for (l_a) are sketched in Fig. 3.

Set $G = G_1 \cup G_2$, where

$$G_1 = \{(x, y) : y \geq 0, -\infty < x < +\infty\},$$

$$G_2 = \{(x, y) : y < 0, -\infty < x < x_1(y)\}.$$

With the same method as above, we also got that in region $G_+ = \{(x, y) : x_2(y) < x < b, |y| < +\infty\}$, $A(x, y, a)M(x, y, a) \leq 0$, and in region $G_- = \{(x, y) : -\infty < x < x_2(y), |y| < +\infty\}$, $A(x, y, a)M(x, y, a) \geq 0$.

In above three cases, we get that $A(x, y, a)M(x, y, a) \leq 0$ for $(x, y) \in G_+$ and $A(x, y, a)M(x, y, a) \geq 0$ for $(x, y) \in G_-$. Applying Theorem B, we conclude that system (l_a) has no multiple limit cycle, in particular system (l_a) has no semistable limit cycle.

Step 3. Prove that (l_a) has at most one limit cycle for any $a \in I$.

Suppose that there exists a parameter $a_0 \geq 0$, such that (l_a) has two limit cycles $\Gamma_2(a_0) \supset \Gamma_1(a_0) \supset O$. As just proved above, $\Gamma_1(a_0)$ and $\Gamma_2(a_0)$ are simple. Without loss of generality, we may assume that there is no limit cycle between

$\Gamma_1(a_0)$ and $\Gamma_2(a_0)$ and there is no other limit cycle in the interior of $\Gamma_1(a_0)$. Let $\lambda(x,y) = \frac{1}{2}(x^2+y^2)$. Then

$$\left.\frac{d\lambda(x,y)}{dt}\right|_{(l_a)} = x\dot{x} + y\dot{y} = -y^2\phi(y)(f(x)-a) \geq 0,\ 0 < |x| \ll 1 .$$

So $O(0,0)$ is a repellor. And since $\Gamma_1(a_0)$ is simple, $\Gamma_1(a_0)$ is stable. It is well known that two adjacent limit cycles possess different stabilities on their adjacent sides. From this fact and the simplicity of $\Gamma_2(a_0)$, we deduce that $\Gamma_2(a_0)$ is simple and unstable. Thus, by the theory of generalized rotated vector fields (See[3], pp. 244-249), if a increases from a_0, then $\Gamma_1(a_0)$ monotonically expands and $\Gamma_2(a_0)$ monotonically contracts. When a reaches some value $a^* > a_0$, $\Gamma_1(a_0)$ and $\Gamma_2(a_0)$ coincide into one semistable limit cycle. This contradicts the fact that (l_{a^*}) has no semistable limit cycle which we just proved above. Therefore, (l_a) has at most one limit cycle for $a \geq 0$, and if it exists, it must be simple and stable.

Before finishing our paper, we discuss the applications of Theorem A and its proof.

Firstly, Theorem A can be applied to the system

$$\ddot{x} + f(x)\dot{x}^{k+1} + g(x) = 0, \tag{2.11}$$

where k is an even number and $f(x)$ and $g(x)$ satisfy the conditions (1) and (2) of Theorem A. Applying Theorem A, we conclude that (2.11) has at most one nontrivial periodic solution and it is stable if it exists.

Н. В. Медведев [12] ever studied the system (2.11) and gave different conditions to guarantee the uniqueness of limit cycle. Besides the condition (1), he assumed

(4) there exist $x_1 < 0 < x_2$ such that $f(x_1) = f(x_2) = 0$, $f(x) < 0$ for all $x \in (x_1, x_2)$, $f(x) > 0$ for all $x \notin [x_1, x_2]$, and $G(x_1) = G(x_2)$.

Under the condition (1) and (4), Sandqvist and Andersen [8] prove that the system

$$\ddot{x} + \mu f(x)|\dot{x}|^{\alpha}\dot{x} + g(x) = 0 \tag{2.12}$$

has at most one limit cycle, where $\mu = 0, -1 < \alpha < +\infty$. (2.12) is equivalent to

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -g(x) - f(x)|y|^{\alpha}y. \end{cases} \tag{2.13}$$

When $-1 < \alpha \leq 1$, the system is not differentiable on $y = 0$. However, checking the proof of Tkachev's theorem in detail, we have found that Tkachev's theorem is still valid if $f(x)$ and $g(x)$ satisfy the conditions (1) and (2). Thus, we conclude that the system (2.13) has at most one limit cycle if the conditions (1) and (2) hold and $-1 < \alpha < +\infty$.

Example. Consider the system

$$\ddot{x} + \mu(x^2 + \gamma x - \beta^2)|\dot{x}|^\alpha \dot{x} + x = 0, \tag{2.14}$$

where $\mu > 0$, γ, $\beta \neq 0$ and $-1 < \alpha < +\infty$. This system is quite simple, but all existing results before cannot be applied to (2.14). However, since

$$\frac{d}{dx}\left(\frac{x^2 + \gamma x - \beta^2}{x}\right) = 1 + \frac{\beta^2}{x^2} > 0$$

for $x \neq 0$, the above discussion shows that this system has a unique stable limit cycle.

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