Algebraic Number Theory
Math 514B Spring 2008

Problem Set 2
Due: Tuesday, Feb. 5th

1. Fix an integer $n > 1$ along with a rational prime p not dividing n. Denote $R = \mathbb{Z}[\zeta_n]$ as the integral closure of $\mathbb{Z}$ in $K = \mathbb{Q}(\zeta_n)$, and $\mathfrak{p}$ as a prime in R lying over p. Assume $p \equiv 1 \bmod n$.

   a) Show that there exists a homomorphism $\chi : (R/\mathfrak{p}R)^* \to (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{C}^*$ such that $\chi^n = 1$.

   b) Show that the map $(\frac{\cdot}{\mathfrak{p}})_n : (R/\mathfrak{p})^* \to \mathbb{C}^*$ defined by $\alpha^{(p-1)/n} \equiv (\frac{\alpha}{\mathfrak{p}})_n \bmod \mathfrak{p}$ is one such homomorphism. (This is the nth-power residue symbol.)

   c) Denote the Gauss sum
   $$S = \sum_{\alpha \in (R/\mathfrak{p})^*} (\frac{\alpha}{\mathfrak{p}})_n \cdot \zeta_p^a$$
   and define a map $\chi : Gal(L/K) \to \mathbb{C}^*$ that sends $\sigma \mapsto \frac{\sigma(S)}{S}$. Show that χ is a homomorphism such that $\chi^n = 1$.

   d) Fix a prime $q = 1 \bmod n$ distinct from p, and choose $\alpha_q \in R/\mathfrak{p}R$ corresponding to $q \in \mathbb{Z}/p\mathbb{Z}$. Show that the Frobenius element corresponding to q maps to
   $$\chi(\sigma_q) = (\frac{\alpha_q}{\mathfrak{p}})_n^{-1}.$$

2. Let K be a real number field of degree $n = r + 2s$ in terms of the number r of real places and the number s of pairs of complex places. We say K has narrow class number 1 if every nonzero fractional ideal is principal and is generated by a totally positive number (i.e. elements $\alpha \in K$ such that $\sigma_i(\alpha) > 0$ for the r real embeddings $\sigma_i : K \hookrightarrow \mathbb{R}$). Show that, with $\mathfrak{m}$ being the modulus which is the product of all of the infinite places of K, the following are equivalent:

   a) K has narrow class number 1;

   b) $\mathbf{C}^{\mathfrak{m}} = \mathbf{I}^{\mathfrak{m}}/i(K_{\mathfrak{m},1})$ is trivial;

   c) The map $K_{\mathfrak{m}}/K_{\mathfrak{m},1} \to \mathbf{C}^{\mathfrak{m}}$ is surjective, and $[\mathbf{U}_K : \mathbf{U}_K \bigcap K_{\mathfrak{m},1}] = 2^r$.

3. Let K be a number field of class number 1. Given a modulus $\mathfrak{m}$, let $\phi : K_{\mathfrak{m}} \to \mathbb{C}^*$ be a multiplicative character that is trivial on $K_{\mathfrak{m},1}$. Show that such a character ϕ factors through a Dirichlet character mod $\mathfrak{m}$ (i.e. $\phi = \chi \cdot i$ in terms of a multiplicative character $\chi : \mathbf{I}^{\mathfrak{m}} \to \mathbb{C}^*$ that is trivial on $i(K_{\mathfrak{m},1})$) if and only if ϕ is trivial on $\mathbf{U}_K$.