Algebraic Number Theory
Math 514B Spring 2008

Problem Set 3
Due: Thursday, Feb. 14th

1. a) Which of the following fields are contained in a cyclotomic field $\mathbb{Q}(\zeta_m)$? For those that are, find the smallest such m: $\mathbb{Q}(\sqrt{35})$, $\mathbb{Q}(\sqrt{-35})$, $\mathbb{Q}(\sqrt[3]{7})$, $\mathbb{Q}(\cos\frac{\pi}{10})$.

   b) Find an abelian extension L of $K = \mathbb{Q}(\sqrt{2})$ that does not lie in $\mathbb{Q}(\zeta_m)$ for any m. (Hint: Let $L = K(\sqrt{\alpha})$, where α is chosen so that L is not Galois over $\mathbb{Q}$. If $L \subset \mathbb{Q}(\zeta_m)$, is $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/L)$ a normal subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$?)

2. Let K be the number field $\mathbb{Q}(\sqrt{15})$, and let the modulus $\mathfrak{m} = \infty_1\infty_2$ be the product of all real primes in K.

   a) Compute the class number $|\mathbf{C}_K|$ and the order of the quotient group $\frac{K_\mathfrak{m}}{K_{\mathfrak{m},1}}$.

   b) Find the unit group $\mathbf{U}_K$ and its subgroup $\mathbf{U}_K \bigcap K_{\mathfrak{m},1}$.

   c) Compute the narrow class group of K (i.e. the ray class group $\mathbf{C}^\mathfrak{m}$ of K for the modulus $\mathfrak{m}$).

3. Consider the extension $K \subset L$ where $K = \mathbb{Q}$ and $L = \mathbb{Q}(i, \sqrt{5})$.

   a) Find the conductor m of L/K and the Artin map

   $$\varphi_{L/\mathbb{Q}} : \mathbf{C}^\mathfrak{m} \longrightarrow \mathrm{Gal}(L/K),$$

   where $\mathbf{C}^\mathfrak{m}$ is the ray class group of K for the modulus $\mathfrak{m} = (m) \cdot \infty$.

   b) Show the Artin map $\varphi_{L/\mathbb{Q}}$ is surjective.

   c) Denote $\mathbf{I}_{L,\mathfrak{m}}$ the kernal of the Artin map $\varphi_{L/\mathbb{Q}}$, explicitly find $\mathbf{I}_{L,\mathfrak{m}}$.