

Algebraic Number Theory
Math 514B Spring 2008

Problem Set 5
Due: Thursday, Feb. 28th

1. Denote the quadratic Dirichlet character $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ in term of the quadratic residue symbol

$$\chi(n) = \left(\frac{-3}{n}\right) = \begin{cases} +1 & \text{if } n = 3k + 1, \\ -1 & \text{if } n = 3k + 2, \\ 0 & \text{if } n = 3k. \end{cases}$$

- a) Show that for each positive integer n for complex s with $Re(s) > 1$,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-ny} y^{s-1} dy \quad \text{and so} \quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{y^{s-1}}{e^y - 1} dy,$$

where we have used the Gamma function $\Gamma(s) = \int_0^\infty -y y^{s-1} dy = (s-1)!$.

- b) Show that for $Re(s) > 1$, we have

$$L(s, \chi) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^y}{1 + e^y + e^{2y}} y^{s-1} dy.$$

- c) Compute the value of $L(s, \chi)$.

2. Define the Mobius function on positive integers as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

- a) Show that for $Re(s) > 1$, we have the Dirichlet series

$$\sum_{n=1}^\infty \frac{\mu(n)}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s}) = \frac{1}{\zeta(s)} \quad \text{and so} \quad \sum_{n=1}^\infty \frac{\mu(n)}{n} = 0.$$

- b) The Mobius Inversion Formula states that for sequences of multiplicative complex numbers $\{a(1), a(2), a(3), \dots\}$ and $\{b(1), b(2), b(3), \dots\}$, we have

$$a(n) = \sum_{d|n} b(d) \iff b(n) = \sum_{d|n} \mu(n/d) \cdot a(d).$$

Show that this is equivalent to the formal identities of Dirichlet series

$$\sum_{n=1}^\infty \frac{a(n)}{n^s} = \zeta(s) \cdot \sum_{n=1}^\infty \frac{b(n)}{n^s} \iff \sum_{n=1}^\infty \frac{a(n)}{n^s} = \frac{1}{\zeta(s)} \cdot \sum_{n=1}^\infty \frac{b(n)}{n^s}.$$

3. Denote K as a number field with ring of integers R . Define the Mobius function for number fields as the map on nonzero integral ideals satisfying

$$\mu(\mathfrak{A}) = \begin{cases} 1 & \text{if } \mathfrak{A} = (1), \\ (-1)^r & \text{if } \mathfrak{A} = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_r \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove the Mobius Inversion Formula for number fields: for sequences of multiplicative complex numbers $\{\cdots, a(\mathfrak{A}), \cdots\}$ and $\{\cdots, b(\mathfrak{A}), \cdots\}$, we have

$$a(\mathfrak{A}) = \sum_{\mathfrak{B}|\mathfrak{A}} b(\mathfrak{B}) \iff b(\mathfrak{A}) = \sum_{\mathfrak{B}|\mathfrak{A}} \mu(\mathfrak{A}\mathfrak{B}^{-1})a(\mathfrak{B}),$$

where the sums are over nonzero integral ideals \mathfrak{B} dividing \mathfrak{A} .