Algebraic Number Theory
Math 514B Spring 2008

Problem Set 6
Due: Thursday, Mar. 6th

1. Let K be a number field, and denote the Mangoldt function as the map $\Lambda_K : \mathbf{I}_K \to \mathbb{C}$ defined on integral ideals by

$$\Lambda_K(\mathfrak{A}) = \begin{cases} \log(\mathcal{N}(\mathfrak{p})) & \text{if } \mathfrak{A} = \mathfrak{p}^e \text{ for some prime } \mathfrak{p} \text{ and integer } e > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that for the Dedekind zeta function we have

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_{\mathfrak{A} \in \mathbf{I}_K \text{integral}} \frac{\Lambda_K(\mathfrak{A})}{(\mathcal{N}(\mathfrak{A}))^s}, \qquad \mathrm{Re}(s) > 1.$$

2. Let $p \equiv 3(4)$ be a prime with $p > 3$, and denote $K = \mathbb{Q}(\sqrt{-p})$. Dirichlet's class number formula states that its class number is

$$|Cl_K| = -\sum_{a=1}^{p-1} (\frac{a}{p}) \cdot \frac{a}{p};$$

where $(\frac{\cdot}{p})$ is the quadratic residue symbol. Use this to show that the class group Cl_K of K has no elements of even order.

3. Let L/K be a finite cyclic Galois extension of degree f, and denote $G = Gal(L/K) = \langle \sigma | \sigma^f = 1 \rangle$ as its group with generator σ. For $a \in L$ nonzero, define the maps $L^* \to L^*$ by

$$N(a) = \prod_{n=0}^{f-1} \sigma^n(a) \qquad \text{and} \qquad \mu(a) = \frac{\sigma(a)}{a}.$$

   a) Show that $H^0(G, L^*) := \ker(\mu)/\mathrm{Im}(N) = K^*/NL^*$; and give an example of an extension L/K such that this quotient is nontrivial.
   b) Show that the following are equivalent:
      i) $H^1(G, L^*) := \ker(N)/\mathrm{Im}(\mu)$ is trivial.
      ii) If $N(a) = 1$ then $a = \sigma(b)/b$ for some $b \in L^*$.