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MATH 517A
HW #3 (Due 11/8)

1) We have $\mathbb{C}(C_2 \times C_2) \cong \mathbb{C}C_4$ as $\mathbb{C}$-algebras.

Proof. Write $C_2 \times C_2 = \{1, a, b, c\}$ and $C_4 = \{1, d, d^2, d^3\}$. Then $\mathbb{C}(C_4) = \mathbb{C}[d]$ is generated by d as a $\mathbb{C}$-algebra, so there is a unique algebra homomorphism $\phi : \mathbb{C}(C_4) \to \mathbb{C}(C_2 \times C_2)$ given by

$$d \mapsto \left(\frac{1}{2} - i\frac{1}{2}\right) \cdot a + \left(\frac{1}{2} + i\frac{1}{2}\right) \cdot b.$$

We claim that ϕ is bijective. Now $\dim_{\mathbb{C}}(\mathbb{C}(C_2 \times C_2)) = 4 = \dim_{\mathbb{C}}(\mathbb{C}(C_4)) = \dim_{\mathbb{C}}(\ker(\phi)) + \dim_{\mathbb{C}}(\mathrm{im}(\phi))$, so it suffices to show injectivity. Given $x \in \mathbb{C}(C_4)$, we may write $x = \alpha_1 \cdot 1 + \alpha_2 \cdot d + \alpha_3 \cdot d^2 + \alpha_4 \cdot d^3$ for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$ since $(1, d, d^2, d^3)$ is a basis. Suppose $\phi(x) = 0$. Now

$$\begin{aligned}
\phi(d^2) &= \phi(d)^2 = \left(\left(\frac{1}{2} - i\frac{1}{2}\right) \cdot a + \left(\frac{1}{2} + i\frac{1}{2}\right) \cdot b\right)^2 \\
&= \frac{1}{4} - i\frac{1}{2} - \frac{1}{4} + \frac{1}{4} + i\frac{1}{2} - \frac{1}{4} + 2\left|\frac{1}{2} + i\frac{1}{2}\right|^2 c = c,
\end{aligned}$$

so

$$\begin{aligned}
\phi(d^3) &= \phi(d)^3 = \phi(d)^2\phi(d) = c\left(\left(\frac{1}{2} - i\frac{1}{2}\right) \cdot a + \left(\frac{1}{2} + i\frac{1}{2}\right) \cdot b\right) \\
&= \left(\frac{1}{2} + i\frac{1}{2}\right) \cdot a + \left(\frac{1}{2} - i\frac{1}{2}\right) \cdot b,
\end{aligned}$$

and

$$\phi(d^4) = (\phi(d^2))^2 = c^2 = 1,$$

whence

$$\begin{aligned}
&0 = \phi(x) = \alpha_1\phi(d) + \alpha_2\phi(d^2) + \alpha_3\phi(d^3) + \alpha_4\phi(d^4) \\
= \ &\alpha_4 \cdot 1 + \left(\alpha_1\left(\frac{1}{2} - i\frac{1}{2}\right) + \alpha_3\left(\frac{1}{2} + i\frac{1}{2}\right)\right) \cdot a + \left(\alpha_1\left(\frac{1}{2} + i\frac{1}{2}\right) + \alpha_3\left(\frac{1}{2} - i\frac{1}{2}\right)\right) \cdot b + \alpha_2 \cdot c
\end{aligned}$$

Thus

$$0 = \alpha_4 = \alpha_1\left(\frac{1}{2} - i\frac{1}{2}\right) + \alpha_3\left(\frac{1}{2} + i\frac{1}{2}\right) = \alpha_1\left(\frac{1}{2} + i\frac{1}{2}\right) + \alpha_3\left(\frac{1}{2} - i\frac{1}{2}\right) = \alpha_2$$

since $(1, a, b, c)$ is a basis for $\mathbb{C}(C_2 \times C_2)$, so

$$\alpha_1\frac{1}{2} + \alpha_3\frac{1}{2} = 0 = \alpha_1\frac{1}{2} - \alpha_3\frac{1}{2},$$

giving

$$\alpha_1 = \alpha_1\frac{1}{2} + \alpha_3\frac{1}{2} + \alpha_1\frac{1}{2} - \alpha_3\frac{1}{2} = 0 + 0 = 0,$$

and similarly

$$\alpha_3 = \alpha_1\frac{1}{2} + \alpha_3\frac{1}{2} - \left(\alpha_1\frac{1}{2} - \alpha_3\frac{1}{2}\right) = 0 - 0 = 0.$$

Therefore $x = 0$, so ϕ is injective, as required. □

2) Each row sum in a character table of a finite group G is a nonnegative integer.

Proof. Let $\chi \in \mathrm{Irr}(G)$ and suppose $K_1, \ldots, K_m$ are the conjugacy classes of G. It suffices to show

$$\sum_{i=1}^{m} \chi(K_i)$$

is a nonnegative integer. Let χ_{adj} be the permutation character corresponding to the action of G on itself by conjugation. Then for each $i = 1, \ldots, m$ we have $k_i \in K_i \Rightarrow$

$$\chi_{\mathrm{adj}}(K_i) = |C_G(k_i)| = \frac{|G|}{|K_i|}.$$

Thus noting that $|K_i| = |K_i^{-1}|$ for all $i = 1, \ldots, m$ we find

$$\begin{aligned} \sum_{i=1}^{m} \chi(K_i) &= \frac{1}{|G|} \sum_{i=1}^{m} |K_i| \chi(K_i) \frac{|G|}{|K_i^{-1}|} = \frac{1}{|G|} \sum_{i=1}^{m} |K_i| \chi(K_i) \chi_{\mathrm{adj}}(K_i^{-1}) \\ &= \frac{1}{|G|} \sum_{i=1}^{m} \sum_{k \in K_i} \chi(k) \chi_{\mathrm{adj}}(k^{-1}) = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi_{\mathrm{adj}}(g^{-1}) \\ &= (\chi, \chi_{\mathrm{adj}}) \in \mathbb{N}_0, \end{aligned}$$

as required. □

3) Let G be a group of odd order and suppose $\chi \in \mathrm{Irr}(G)$ is not the trivial character. Then $\chi \neq \bar{\chi}$.

Proof. First, define a relation $\sim$ on $G \backslash \{1\}$ by $g \sim h \Leftrightarrow g = h$ or $g = h^{-1}$. Then $\sim$ is an equivalence relation and each equivalence class consists of exactly two elements since if $g \in G$ and $g = g^{-1}$, then $g^2 = 1$, so $1 = |1| = |g^2| = |g|/(|g|, 2) = |g|$ because $|g|$ is odd, giving $g = 1$. Thus $|G \backslash \{1\}| = 2m$ where m is the number of equivalence class under $\sim$. Choosing equivalence class representatives $r_1, \ldots, r_m \in G \backslash \{1\}$, we have that $G \backslash \{1\}$ is the disjoint union $\{r_1, \ldots, r_m\} \dot{\cup} \{r_1^{-1}, \ldots, r_m^{-1}\}$. Now suppose $\chi = \bar{\chi}$; we will obtain a contradiction. Then since χ is not the trivial character, we find

$$0 = |G| \cdot 0 = |G| (\chi, \chi_{\mathrm{triv}}) = \sum_{g \in G} \chi(g),$$

so

$$\begin{aligned} \chi(1) &= -\sum_{g \in G \backslash \{1\}} \chi(g) = -\sum_{i=1}^{m} \chi(r_i) - \sum_{i=1}^{m} \chi(r_i^{-1}) \\ &= -\sum_{i=1}^{m} \chi(r_i) - \sum_{i=1}^{m} \bar{\chi}(r_i) = -2 \sum_{i=1}^{m} \chi(r_i). \end{aligned}$$

Next, $\chi(r_i)$ is an algebraic integer for all $i = 1, \ldots, m$, so

$$r := -\sum_{i=1}^{m} \chi(r_i)$$

is an algebraic integer, but $r = \chi(1)/2 \in \mathbb{Q}$, whence $r \in \mathbb{Z}$. Therefore $2 | \chi(1)$, but $\chi(1) \big| |G|$, so $2 \big| |G|$, which is a contradiction again since $|G|$ is odd. □

4) Let G be a finite group and let $g \in G$.

a) Fix $x \in G$. Then g is conjugate in G to $[x, y]$ for some $y \in G$ if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(x)|^2 \overline{\chi(g)}}{\chi(1)} \neq 0.$$

Proof. Let $K_1, \ldots, K_r$ be the conjugacy classes of G with $x \in K_j$, $K_i = K_j^{-1}$, and $g \in K_l$. Then the formula for computing the structure constants from the last homework shows that

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(x)|^2 \overline{\chi(g)}}{\chi(1)} = \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x^{-1})\chi(x)\overline{\chi(g)}}{\chi(1)} = n_{ijl} \frac{|G|}{|K_i||K_j|},$$

so the sum is nonzero if and only if $n_{ijl} \neq 0$. If $n_{ijl} \neq 0$, then there are $k_i \in K_i$, $k_j \in K_j$ such that $k_i k_j = g$, but $k_i = a^{-1}x^{-1}a$ and $k_j = b^{-1}xb$ for some $a, b \in G$, so taking $y = ba^{-1}$, we find

$$g = a^{-1}x^{-1}ab^{-1}xb = a^{-1}(x^{-1}ab^{-1}xba^{-1})a = a^{-1}(x^{-1}y^{-1}xy)a = a^{-1}[x, y]a,$$

showing that g is conjugate to $[x, y]$. Conversely, suppose g is conjugate to $[x, y]$ for some $y \in G$. Then there is a $c \in G$ such that $x^{-1}(y^{-1}xy) = [x, y] = c^{-1}gc \in K_l$ with $x^{-1} \in K_i$ and $y^{-1}xy \in K_j$, so $n_{ijl} = |\{(\alpha, \beta) \in K_i \times K_j | \alpha\beta = g\}| > 0$. $\square$

b) We have $g = [x, y]$ for some $x, y \in G$ if and only if

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0.$$

Proof. First, we note that

$$\begin{aligned} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \quad &\neq \quad 0 \Leftrightarrow 0 \neq \overline{\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}} = \sum_{\chi \in \mathrm{Irr}(G)} \frac{\overline{\chi(g)}}{\chi(1)} = \sum_{\chi \in \mathrm{Irr}(G)} \frac{(\chi, \chi)\overline{\chi(g)}}{\chi(1)} \\ &= \quad \sum_{\chi \in \mathrm{Irr}(G)} \sum_{a \in G} \frac{\chi(a)\chi(a^{-1})\overline{\chi(g)}}{\chi(1)} = \sum_{a \in G} \sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(a)|^2 \overline{\chi(g)}}{\chi(1)}. \end{aligned}$$

where we have used the first orthogonality relations and the fact that $\chi(1) \in \mathbb{N} \subseteq \mathbb{R}$ for all $\chi \in \mathrm{Irr}(G)$. In fact, for each $a \in G$, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(a)|^2 \overline{\chi(g)}}{\chi(1)} = n \frac{|G|}{|\mathrm{cl}_G(a)|^2} \geq 0$$

where n is a structure constant, so

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0 \Leftrightarrow \sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(a)|^2 \overline{\chi(g)}}{\chi(1)} \neq 0 \Leftrightarrow g \in \mathrm{cl}_G([a, b])$$

for some $a, b \in G$ by part a) above, but $[a, b]^c = [a^c, b^c] \ \ \forall c \in G$, giving

$$\sum_{\mathrm{Irr}(G)} \frac{|\chi(x)|^2 \overline{\chi(g)}}{\chi(1)} \neq 0 \Leftrightarrow g = [x, y]$$

for some $x, y \in G$, $\square$

c) All elements of A_5 are commutators. Also, there is a group where there are elements in the commutator subgroup that are not commutators.

Proof. Next, I used the following commands (based upon part b above) in GAP to define a function `f` on finite groups `g` which returns the number of conjugacy classes `K` of `g` such that `K` is contained in the derived subgroup of `g` and no element of `K` is a commutator (equivalently some element of `K` is not a commutator):

```
gap> f:=function(g)
> local i,d,n,irr;
> d:=ClassPositionsOfDerivedSubgroup(CharacterTable(g));
> n:=0;irr:=Irr(g);
> for i in d do
> if Sum(irr,x->x[i]/x[1])=0 then
> n:=n+1;
> fi;
> od;
> return n;
> end;;
```

Now A_5 is simple and nonabelian, so $[A_5, A_5] = A_5$. Thus to prove that every element of A_5 is a commutator it sufficed to compute

```
gap> f(AlternatingGroup(5));
0
```

I checked finite groups of increasing order using the following while loop to find the least `n` such that there exists a group `g` whose derived subgroup is not the equal to set of commutators:

```
gap> i:=0;;n:=1;;
gap> while i=0 do
> a:=AllSmallGroups(n);
> for j in [1..Length(a)] do
> if f(a[j])>0 then
> i:=1;
> g:=a[j];
> fi;
> od;
> n:=n+1
> od;
gap> g;
<pc group of size 96 with 6 generators>
```

Here `g` is a group in the list `AllSmallGroups(96)` such that `f(g) > 0` (in fact, `f(g)=1`). In particular, there are groups where the commutator subgroup is not equal to the set of commutators; moreover, the least order of such a group is 96. □