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ATTRACTIVE N-TYPE CONTACT PROCESSES

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Interacting particle systems are of interest for the purposes of creating spatially explicit models in various disciplines. Monotonicity, also called attractiveness, is one of the basic properties explored for these processes. Its importance is underlined by the ability for sampling exactly from the stationary distribution of an ergodic monotone process with a finite state space using coupling from the past. Monotonicity is well understood for spin systems such as the contact process. Spin systems however only include two particle types while in many applied models, it is desired to include more species of particles. In this paper a general framework of monotonicity will be outlined for a large class of models with contact interactions.

1. Introduction. Interacting particle systems are Markov processes which are used for creating various types of models. The contact process and voter model are two of the most well known interacting particles systems and are 2 processes in a class known as *spin systems*. Creating models for studying competition of species and spread of infections, among others, are applications for which these two processes are most often used. However, the basic models only include two possible states for each site, and it is often the case that applied models will include more than 2 particle types.

Monotonicity and coupling are two of the basic properties studied which often simplify calculations and emphasize behaviors of the process. Coupling from the past(CFTP), introduced by Propp and Wilson in [10], is an algorithm for sampling exactly from the stationary distribution of an irreducible and aperiodic Markov chain with a finite state space. If the process is monotone and has extremal states $\bar{0}$ and $\bar{1}$, then CFTP only requires these states as input. If the condition of monotonicity is not satisfied, it is required to run the algorithm from all states. The state space can however be large which makes this difficult. The CFTP algorithm will not be discussed here although it is one of the primary motivations for this work. Monotonicity is well understood for spin systems, but when multiple particle types are introduced it is not clear what is generally required for monotonicity to be present. This paper presents a general framework to determine monotonicity of multi-type interacting particle systems whose interactions are built from those of the basic contact process.

2. Contact Processes. The basic contact process is a spin system. The class of spin systems encompasses interacting particle systems with two particle types, usually $S = \{0, 1\}$. These particle values are usually distributed on the $d-$dimensional integer lattice, $\mathbb{Z}^d$. The state space is then the metric space, $\Xi = \{0, 1\}^{\mathbb{Z}^d}$, and will also be referred to as configuration space. A state or configuration of particles, $\eta \in \Xi$, is then an infinite array of 0's and 1's. The transition rates are given in terms of a *flip rate*, $c(x, \eta)$, which gives the exponential rate at which a site changes its particle type.

The contact process is the main spin system of interest here and has the following transition

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rates:

$$\begin{array}{cc} \text{transition} & \text{rate} \\ 0 \to 1 & \lambda n_1 \\ 1 \to 0 & \delta \end{array}$$

where n_1 is the number of nearest neighbors occupied by 1's. Infection occurs at an exponential rate proportional to the number of infected neighbors. Thus $c(x, \eta) = \lambda \sum_{|x-y|=1} I_{\{1\}}(\eta(y))$ when $\eta(x) = 0$, and $c(x, \eta) = \delta$ when $\eta(x) = 1$. If $\eta(x) = 1$, then it waits until its exponential clock goes off at rate δ, then it becomes a 0, and if $\eta(x) = 0$, and for example, has 3 occupied neighbors, then this site waits for an exponential clock to chime at rate 3λ at which point it becomes a 1. Generally $\delta = 1$, fixing the timescale.

Numerous variations have been created around the rule that infection occurs at a rate proportional to the number of infected neighbors. The basic two-type contact process developed by Neuhauser[9] is a contact process with the inclusion of another species. which has the following transition rules:

$$\begin{array}{cccc} 0 \to 1 & \lambda_1 n_1 & 0 \to 2 & \lambda_2 n_2 \\ 1 \to 0 & \delta_1 & 2 \to 0 & \delta_2 \end{array}$$

The two-stage contact process studied by Krone[6] is another two-type contact process which is for a species that has multiple life stages. The transitions are that 2's give birth to 1's and the 1's mature into 2's at a constant rate:

$$\begin{array}{cccc} 0 \to 1 & \lambda n_2 & 1 \to 2 & \gamma \\ 1 \to 0 & 1 + \delta & 2 \to 0 & 1 \end{array}$$

The grass–bushes–trees successional model proposed by Durrett and Swindle[2] is a two-type contact process as well. It is the basic two-type contact process when 2's are allow to give birth onto 1's at the same rate that they give birth onto empty sites:

$$\begin{array}{cccccc} 0 \to 1 & \lambda_1 n_1 & 0 \to 2 & \lambda_2 n_2 & 1 \to 2 & \lambda_2 n_2 \\ 1 \to 0 & \delta_1 & 2 \to 0 & \delta_2 & & \end{array}$$

These models hold in common the idea that transitions occur at either a constant rate or at rates proportional to the number of neighbors of a certain type. However they are no longer spin systems so a common understanding of the structure of monotonicity for these types of models is desirable. The reader must note that from now on we will refer to the two-type contact process as a model with three particle types; we give the total number of possible values assigned to a site.

3. Monotonicity. A Feller process with a partially ordered state space, Ξ, and semigroup, $S(t)$, is said to be *monotone* if either of the equivalent conditions

$$f \in \mathscr{M} \text{ implies } S(t)f \in \mathscr{M} \text{ for all } t \geq 0 \tag{3.1a}$$

$$\mu_1 \leq \mu_2 \text{ implies } \mu_1 S(t) \leq \mu_2 S(t) \text{ for all } t \geq 0 \tag{3.1b}$$

is satisfied. Where the μ_i are probability measures on the state space, and $\mathscr{M}$ is the set of continuous monotone functions, $f : \Xi \to \mathbb{R}$, such that $\eta \leq \xi$ implies $f(\eta) \leq f(\xi)$. For the proof of equivalency, see Liggett IPS[7]. Furthermore, $\mu_1 \leq \mu_2$ is equivalent to there existing a measure, ν on $\Xi \times \Xi$, that satisfies

(a) $\nu\{(\eta, \xi) : \eta \in A\} = \mu_1(A)$, and
(b) $\nu\{(\eta, \xi) : \xi \in A\} = \mu_2(A)$, where A is any Borel set in Ξ, and
(c) $\nu\{(\eta, \xi) : \eta \leq \xi\} = 1$.

In other words, the ability to construct a coupling that preserves the partial ordering a.s. is equivalent to the process being monotone. For the proof of this, see Liggett IPS as well. For interacting particle systems, a monotone process is usually called attractive. We want to understand here, what it means for multi-type contact processes to be attractive.

3.1. *Attractive Spin Systems.* For a spin system to be attractive, the flip rate only needs to satisfy the following conditions for $\eta \leq \xi$.

$$c(x,\eta) \leq c(x,\xi) \quad \text{when} \quad \eta(x) = \xi(x) = 0, \tag{3.2a}$$
$$c(x,\eta) \geq c(x,\xi) \quad \text{when} \quad \eta(x) = \xi(x) = 1. \tag{3.2b}$$

This means that given any state, replacing some of the 0's in the configuration of particles by 1's can only maintain or increase the flip rate of 0's, and should maintain or decrease the flip rate for 1's. The proof that this is equivalent to a spin system being monotone is fairly short and can be found in Liggett IPS. For a multi-type process, however, it is a bit more complicated.

3.2. *Attractive Multi-Type Contact Processes.* Monotonicity is seldom discussed for interacting particle systems with three or more types of particles. The two-stage contact process was proved to be attractive in [6], and in [1], Durrett and Neuhauser discussed a certain monotonicity property of a 4-type competition model.

In [6], Krone proves that the two stage contact process is attractive and has some monotonicity properties with respect to its transition parameters in the following theorem.

THEOREM 3.1. (Thm. 3.1 [6]) *Given two sets of parameters for the two-stage contact process satisfying $\lambda_1 \leq \lambda_2$, $\gamma_1 \leq \gamma_2$, and $\delta_1 \geq \delta_2$, then the processes $\xi_t^{(1)}$ and $\xi_t^{(2)}$ with the parameters $(\lambda_1, \gamma_1, \delta_1)$ and $(\lambda_2, \gamma_2, \delta_2)$ respectively, can be constructed on a common probability space so that $\xi^{(1)} \leq \xi^{(2)} \implies \xi_t^{(1)} \leq \xi_t^{(2)}$*

In [1], a similar result is proven for a model with transitions described as follows.

$$\begin{array}{llll}
0 \to 1 & \beta_1(f_1 + c_1 f_3) & 1 \to 0 & \delta_1 \\
0 \to 2 & \beta_2(f_2 + c_2 f_3) & 2 \to 0 & \delta_2 \\
2 \to 3 & c_1\beta_1(f_1 + c_1 f_3) & 3 \to 2 & \delta_1/c_1 \\
1 \to 3 & c_2\beta_2(f_2 + c_2 f_3) & 3 \to 1 & \delta_2/c_2
\end{array}$$

THEOREM 3.2. (Prop. 1.1 [1]) *Let ξ_t and ξ_t' be the processes with transitions described above and have parameters $(\beta_1, \beta_2, \delta_1, \delta_2, c_1, c_2)$ and $(\beta_1', \beta_2', \delta_1', \delta_2', c_1', c_2')$ such that $\beta_1 \leq \beta_1', \beta_2 \geq \beta_2', \delta_1 \geq \delta_1', \delta_2 \leq \delta_2', c_1 \leq c_1', c_2 \geq c_2'$. Let $\eta_t = \{x : \xi_t(x) = 1 \text{ or } 3\}$ and $\zeta_t = \{x : \xi_t(x) = 2 \text{ or } 3\}$, and define η_t' and ζ_t' accordingly for ξ_t'. If the initial conditions satisfy $\eta \subset \eta'$ and $\zeta \supset \zeta'$, then the processes (η_t, ζ_t) and (η_t', ζ_t') can be constructed on a common probability space so that $\eta_t \subset \eta_t'$ and $\zeta_t \supset \zeta_t'$ for all $t \geq 0$.*

So this process is not attractive in the usual sense but is attractive when considering the set-valued states, η and ζ.

Later on it will become clear how similar statements can be made about arbitrary attractive n-type contact processes. What follows below is a definition of attractiveness for a class of n-type contact processes in term of the transition rates similar to 3.2a and 3.2a for spin systems. However, since more complicated transitions are allowed, we need to develop a way to organize the interactions. For this, what is called the *interaction map* is introduced.

4. The Particle Interaction Map.

The foundation of this approach is the use of what is called the *particle interaction map*. An interaction is defined in terms of two particles and is allowed to result in a change to exactly one of the interacting particles. We start with the finite set of totally ordered particle types, $S = \{0, 1, 2, \ldots, n\}$.

DEFINITION 5. *Given a finite set of totally ordered particle values, S, the map, $\mathscr{I} : S \times S \to S$ is called a particle interaction map if its domain is all of $S \times S$.*

Having two particles, b and a, as input, the interaction map gives the resulting effect on particle a. So $\mathscr{J}(b,a)$ is the resulting influence on particle type a by type b. By convention, b is called the neighboring particle. The domain of $\mathscr{J}$ is partitioned into sets of *up*, *null*, and *down* interactions: $\mathcal{U} = \{(b,a) \in S \times S | \mathscr{J}(b,a) > a\}$, $\mathcal{N} = \{(b,a) \in S \times S | \mathscr{J}(b,a) = a\}$, and $\mathcal{D} = \{(b,a) \in S \times S | \mathscr{J}(b,a) < a\}$. These represent interactions which result in larger, unchanged, and lower particle values respectively.

Figure 1 shows the interaction maps for two well studied spin systems. Although, there is no need to formulate these processes in terms of the interaction maps since the types of interactions allowed are limited to single possibilities in any case: there is only one possible change for each particle type since there are only two particle types in total. When we expand to the multi-type contact process, this formulation becomes more useful.

particle affected by the neighbor

neighbor

$_b\backslash^a$	0	1
0	0	0
1	1	0

(C)

$_b\backslash^a$	0	1
0	0	0
1	1	1

(V)

FIG 1. *Interaction Maps for the Contact Process, (C), and Voter model, (V).*

$_b\backslash^a$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

FIG 2. *Interaction map formulation for the multi-type contact process*

It may at first seem unnatural (except for the 0 particle) for each particle type in the multi-type contact process to interact with a 1 or 2 causing them to become 0's, but this is due to there being constant death rates. When a 1 or 2 is sitting at a site, since the rates for independent exponential random variables sum and the only transition is to become a 0, we get our constant death rates.

DEFINITION 6. *The interaction map, $\mathscr{J}$, is called* non-decreasing *on $A \subset S \times S$ if for any two (b_1, a_1) and (b_2, a_2) in A which satisfy $(b_1, a_1) \leq (b_2, a_2)$ (meaning $b_1 \leq b_2$ and $a_1 \leq a_2$), it follows that $\mathscr{J}(b_1, a_1) \leq \mathscr{J}(b_2, a_2)$.*

DEFINITION 7. *The particle interaction map will be called* attractive *if it satisfies the following conditions:*

(a) $\mathscr{J}$ is non-decreasing on $\mathcal{U}$.
(b) $\mathscr{J}$ is non-decreasing on $\mathcal{D}$.
(c) If $(b_1, a_1) \leq (b_2, a_2)$, $(b_1, a_1) \in \mathcal{U}$, and $(b_2, a_2) \in \mathcal{D} \cup \mathcal{N}$, then $\mathscr{J}(b_1, a_1) \leq a_2$.
(d) If $(b_1, a_1) \leq (b_2, a_2)$, $(b_1, a_1) \in \mathcal{U} \cup \mathcal{N}$, and $(b_2, a_2) \in \mathcal{D}$, then $a_1 \leq \mathscr{J}(b_2, a_2)$.

Notice that the last two conditions of Definition 7 can be restated as follows

(c′) If $(b_1, a_1) \in \mathcal{U}$, then $\mathscr{J}(b_1, a_1) \leq \min_a \{a : (b,a) \in \mathcal{D} \cup \mathcal{N}\}$.
(d′) If $(b_2, a_2) \in \mathcal{D}$, then $\max_a \{a : (b,a) \in \mathcal{U} \cup \mathcal{N}\} \leq \mathscr{J}(b_2, a_2)$.

The purpose of this definition is that if two ordered configurations undergo simultaneous interactions, the partial ordering will be preserved. It will later be seen to be a requirement for the process to be monotone. The first two conditions state that when considering an interaction between two sites, if the particle type is increased or decreased at one of the sites and still has the possibility of an up or down transition respectively, the resulting transitions from the particle interactions should preserve this order. The third condition states that if a configuration is such that an up transition is possible for a particular site and its neighbor, then the resulting replacement particle cannot be above that for any higher configuration for which the possible transition is null or down. The last condition states that if a configuration allows a down transition for a particular site and its neighbor, then the replacement particle cannot be below that for any lower configuration having the only possible transition as null or up.

The following definitions tells us about the ability for interactions to change one particle type into another. This is necessary to consider the possibility of creating an irreducible process. We want to know what particle types can communicate.

DEFINITION 8. *Given some interaction map,* $\mathscr{I}$*, particle value* b *is said to be* first reachable *by particle value* a*, if there exists a particle value* $\tilde{a}$ *such that* $\mathscr{I}(\tilde{a}, a) = b$*.*

DEFINITION 9. *Given an interaction map,* $\mathscr{I}$*, particle* b *is said to be* accessible *by particle* a*, if there exist particle values* $a_1, a_2, ..., a_n$ *such that* a_1 *is first reachable by* a*,* a_{k+1} *is first reachable by* a_k *for* $k = 1, ..., n-1$*, and* b *is first reachable by* a_n

DEFINITION 10. *Given some interaction map,* $\mathscr{I}$*, two particle values,* a *and* b*, are said to* communicate *if they are both accessible by each other.*

DEFINITION 11. *The particle interaction map will be called* irreducible *if all particles communicate.*

If we have an irreducible interaction map, we would like to build an irreducible process from it. An irreducible interaction map does not guarantee an irreducible process however. Just as with the absorbing all 0's state of the contact process, we may have absorbing states or classes. Irreducibility of the interaction map only guarantees that any particle can eventually be replaced by any other given a proper external configuration of neighboring particles.

For the purposes of creating models where only finite grids are considered, the process can usually be made irreducible by including boundary conditions or spontaneous birth rates. Boundary conditions will not affect monotonicity, but spontaneous birth rates may. This will be discussed in more detail later.

11.1. *Constructing Attractive Interaction Maps.*

In order to understand monotonicity for multi-type contact processes it is helpful to start with determining what kinds on interactions are allowed in an attractive process. To understand the properties of an attractive interaction map is a little difficult from merely reading the definition. The following lemmas should provide more insight. What we want to achieve is a rigorous development of the mathematical structure resulting from Definition 7.

LEMMA 12. *If* $(b, a) \in \mathcal{D}$*, for an attractive interaction map* $\mathscr{I}$*, then for any* $b_1 < b$*,* $(b_1, a) \in \mathcal{D}$ *as well. Similarly, if* $(b, a) \in \mathcal{U}$*, then for any* $b_2 > b$*,* $(b_2, a) \in \mathcal{U}$*.*

PROOF. Assume that $(b, a) \in \mathcal{D}$ and that $(b-1, a) \in \mathcal{U} \cup \mathcal{N}$. By condition (4) in the definition of an attractive interaction map, $\mathscr{I}(b, a) \geq a$. However this is contradictory since it is a strict down transition.

If we assume that $(b,a) \in \mathcal{U}$ and that $(b+1,a) \in \mathcal{D} \cup \mathcal{N}$, then by condition (3) of definition 7 we have $\mathscr{J}(b,a) \leq a$. This again is a contradiction. We have only shown the result for changing the neighbor's particle value by one, but the rest follow by induction. □

Theorem 12.1. *The interaction map $\mathscr{J}$ is attractive if and only if*

(a) $\mathscr{J}$ is non-decreasing on $\mathcal{U} \cup \mathcal{N}$ and non-decreasing on $\mathcal{D} \cup \mathcal{N}$,

(b) If $\exists b, a \in S$, such that $(b,a) \in \mathcal{U}$, and $(b,a+1) \in \mathcal{D}$, then $\mathscr{J}(b,a) = a+1$ and $\mathscr{J}(b,a+1) = a$.

Proof. First assume we have an attractive interaction map, $\mathscr{J}$. Take $(b_1,a_1) \leq (b_2,a_2)$ and both are in $\mathcal{U} \cup \mathcal{N}$. Because $\mathscr{J}$ is non-decreasing on $\mathcal{U}$, the only issue is when $(b_1,a_1) \in \mathcal{U}$ and $(b_2,a_2) \in \mathcal{N}$. But due to (3) of definition 7, $\mathscr{J}(b_1,a_1) \leq a_2$. Similarly, if $(b_1,a_1) \in \mathcal{N}$ and $(b_2,a_2) \in \mathcal{D}$, $\mathscr{J}(b_2,a_2) \geq a_1$. Together these prove that (1) in the statement of the theorem holds.

For the second statement of the theorem, note that $a < \mathscr{J}(b,a) \leq a+1$ and $a \leq \mathscr{J}(b,a+1) < a+1$ must be true due to the third and fourth conditions of Definition 7. Then it holds that $\mathscr{J}(b,a) = a+1$ and $\mathscr{J}(b,a+1) = a$.

Now we assume that the statements of the theorem are satisfied and show that the conditions of Definition 7 follow. Beginning with statement (1) of the theorem, the first two conditions of Definition 7 directly follow since $\mathcal{U}$ and $\mathcal{D}$ are subsets of $\mathcal{U} \cup \mathcal{N}$ and $\mathcal{D} \cup \mathcal{N}$ respectively.

Next, take arbitrary $(b_1,a_1) \in \mathcal{U}$ and $(b_2,a_2) \in \mathcal{D} \cup \mathcal{N}$ with $(b_1,a_1) \leq (b_2,a_2)$, and define $\tilde{a} = \max\{a : (b_1,a_1) \leq (b_2,a), a < a_2, (b_2,a) \in \mathcal{U}\}$. By this, it should be clear that $(b_2,\tilde{a}+1) \in \mathcal{D} \cup \mathcal{N}$ and $\tilde{a}+1 \leq a_2$. Then the next steps follow:

$$\begin{aligned} &\mathscr{J}(b_1,a_1) \leq \mathscr{J}(b_2,\tilde{a}), \text{By (1) of the theorem statement} \\ &\text{and } \mathscr{J}(b_2,\tilde{a}+1) \leq \tilde{a}+1 \leq a_2. \end{aligned}$$

If $(b_2,\tilde{a}+1) \in \mathcal{D}$, then by statement (2) of the theorem, $\mathscr{J}(b_2,\tilde{a}) = \tilde{a}+1$, and $\mathscr{J}(b_2,\tilde{a}+1) = \tilde{a}$. Then $\mathscr{J}(b_1,a_1) \leq \mathscr{J}(b_2,\tilde{a}) = \tilde{a}+1 \leq a_2$ because (b_1,a_1) and $(b_2,\tilde{a})$ are contained in $\mathcal{U}$.

If $(b_2,\tilde{a}+1) \in \mathcal{N}$, then $\mathscr{J}(b_1,a_1) \leq \mathscr{J}(b_2,\tilde{a}+1) = \tilde{a}+1 \leq a_2$ by statement (1) of the theorem since both (b_1,a_1) and $(b_2,\tilde{a}+1)$ are members of $\mathcal{U} \cup \mathcal{N}$.

We conclude that $\mathscr{J}(b_1,a_1) \leq a_2$ satisfying (3) of Definition 7. Now, take arbitrary $(b_1,a_1) \in \mathcal{U} \cup \mathcal{N}$ and $(b_2,a_2) \in \mathcal{D}$ with $(b_1,a_1) \leq (b_2,a_2)$, and define $\tilde{a} = \min\{a : (b_1,a) \leq (b_2,a_2), a_1 < a, (b_1,a) \in \mathcal{D}\}$. By this, it should be clear that $(b_1,\tilde{a}-1) \in \mathcal{U} \cup \mathcal{N}$ and $a_1 \leq \tilde{a}-1$. The following inequalities then hold:

$$\begin{aligned} &\mathscr{J}(b_1,a_1) \leq \mathscr{J}(b_1,\tilde{a}-1), \text{By (1) of the theorem statement} \\ &\text{and } a_1 \leq \tilde{a}-1 \leq \mathscr{J}(b_1,\tilde{a}-1). \end{aligned}$$

If $(b_1,\tilde{a}-1) \in \mathcal{U}$, then by statement (2) of the theorem, $\mathscr{J}(b_1,\tilde{a}) = \tilde{a}-1$, and $\mathscr{J}(b_1,\tilde{a}-1) = \tilde{a}$. Then $a_1 \leq \tilde{a}-1 = \mathscr{J}(b_1,\tilde{a}) \leq \mathscr{J}(b_2,a_2)$ because $(b_1,\tilde{a})$ and (b_2,a_2) are are contained in $\mathcal{D}$.

If $(b_1,\tilde{a}-1) \in \mathcal{N}$, then $a_1 \leq \tilde{a}-1 = \mathscr{J}(b_1,\tilde{a}-1) \leq \mathscr{J}(b_2,a_2)$ by statement (1) of the theorem since both $(b_1,\tilde{a}-1)$ and (b_2,a_2) are members of $\mathcal{D} \cup \mathcal{N}$.

We conclude that $a_2 \leq \tilde{a}-1 \leq \mathscr{J}(b_2,a_2)$ satisfying (4) of Definition 7. This proves that the definition of attractive is equivalent to the statements given by Lemma 12.1. □

Theorem 12.1 gives us a more intuitive picture of what an attractive interaction map actually looks like. When going directly from the set $\mathcal{U}$ to $\mathcal{D}$ horizontally(in terms of the table formats given), there is exactly a decrease of one. This can be seen in Figure 3 below. In example (a) $\mathscr{J}(2,1) = 2$, an up transition, and $\mathscr{J}(2,2) = 1$, a down transition, so we can see that this satisfies Theorem 12.1.

LEMMA 13. *If* $\mathscr{J}$ *is an attractive interaction map,* $a_1 < a_2$*, and for some particle,* b*,* $\mathscr{J}(b, a_1) > \mathscr{J}(b, a_2)$*, then* $(b, a_1) \in \mathcal{U}$ *and* $(b, a_2) \in \mathcal{D}$*.*

PROOF. If $(b, a_1) \in \mathcal{D} \cup \mathcal{N}$, then the interaction map cannot be defined for a_2. If $a_2 \in \mathcal{D} \cup \mathcal{N}$, there is a problem because the attractive interaction map is non-decreasing on $\mathcal{D} \cup \mathcal{N}$ and $a_1 < a_2$ would mean that $\mathscr{J}(b, a_1) \leq \mathscr{J}(b, a_2)$. We also find that (b, a_2) is not a member of $\mathcal{U}$ since we get the inequality $a_2 > a_1 \geq \mathscr{J}(b, a_1) > \mathscr{J}(b, a_2)$ by the statement of the lemma when assuming $(b, a_1) \in \mathcal{D} \cup \mathcal{N}$. From this we see that $\mathscr{J}(b, a_2) > a_2$ as well if $(b, a_2) \in \mathcal{U}$. So $(b, a_1) \in \mathcal{U}$ is the only possibility. Now (b, a_2) must be a member of $\mathcal{D}$ because $\mathscr{J}$ is non-decreasing on $\mathcal{U} \cup \mathcal{N}$, and $a_1 < a_2$ would have to give $\mathscr{J}(b, a_1) \leq \mathscr{J}(b, a_2)$ which is contrary to the assumption of the lemma. □

LEMMA 14. *If* $\mathscr{J}$ *is an attractive interaction map,* $(b, a_1) \in \mathcal{U} \cup \mathcal{N}$*,* $(b, a_2) \in \mathcal{D} \cup \mathcal{N}$*, and* $a_1 < a_2$*, and* $\mathscr{J}(b, a_1) > \mathscr{J}(b, a_2)$*, then define* $\tilde{a} = \mathscr{J}(b, a_1)$ *and the following equations hold:*

$$\text{For } a \in [a_1..\tilde{a} - 1],\ \mathscr{J}(b, a) = \tilde{a} \tag{14.1a}$$

$$\text{For } a \in [\tilde{a}..a_2],\ \mathscr{J}(b, a) = \tilde{a} - 1 \tag{14.1b}$$

PROOF. By lemma 13, $(b, a_1) \in \mathcal{U}$ and $(b, a_2) \in \mathcal{D}$. Combining this with the fact that this interaction map is attractive we also have, by properties (3) and (4) of definition 7, $\mathscr{J}(b, a_1) \leq a_2$ and $a_1 \leq \mathscr{J}(b, a_2)$. Choose an arbitrary $a \in [a_1..\tilde{a} - 1]$. If $(b, a) \in \mathcal{D} \cup \mathcal{N}$, then $a_1 < \tilde{a} = \mathscr{J}(b, a_1) \leq a \leq \tilde{a} - 1$ which is contradictory so $(b, a) \in \mathcal{U}$. Now take $a \in [\tilde{a}..a_2]$. Suppose $(b, a) \in \mathcal{U} \cup \mathcal{N}$. This would give $a_2 \geq \tilde{a} = \mathscr{J}(b, a_1) \geq a > \tilde{a} - 1$ which is a contradiction as well, so $(b, a) \in \mathcal{D}$.

For $a \in [a_1..\tilde{a} - 1]$, $\tilde{a} = \mathscr{J}(b, a_1) \leq \mathscr{J}(b, a) \leq \tilde{a}$ because $\tilde{a} \in \mathcal{D}$, so $\mathscr{J}(b, a) = \tilde{a}$. For $a \in [\tilde{a}..a_2]$, $\tilde{a} - 1 \leq \mathscr{J}(b, a) \leq \mathscr{J}(b, a_2) < \mathscr{J}(b, a_1) = \tilde{a}$ because $\tilde{a} - 1 \in \mathcal{U}$, so $\mathscr{J}(b, a) = \tilde{a} - 1$. This completes the proof. □

In other words, if the conditions for this lemma are satisfied, then the interaction map is a step function from one value to the other across the interval $[a_1..a_2]$ for fixed b.

THEOREM 14.1. *The interaction map* $\mathscr{J}$ *is attractive if and only if given any* $(b, a) \in S \times S$*,* $\mathscr{J}(b, a) \leq \mathscr{J}(b + 1, a)$ *and only one of the following is satisfied:*

(a) $\mathscr{J}(b, a) \leq \mathscr{J}(b, a + 1)$*.*
(b) $\mathscr{J}(b, a) = a + 1$ *and* $\mathscr{J}(b, a + 1) = a$*.*

PROOF. Suppose the map is attractive, then $\mathscr{J}(b, a) \leq \mathscr{J}(b + 1, a)$ by Lemma 12. If (b, a) and $(b, a + 1)$ are both in $\mathcal{U}$ or both in $\mathcal{D}$, then (a) above is satisfied. If they are in $\mathcal{U}$ and $\mathcal{D}$ respectively, then (b) is satisfied by Theorem 12.1. Including the set $\mathcal{N}$ into the above arguments does not introduce any complications.

Now suppose the conditions of this theorem are satisfied, then $(b, a) \leq (b_1, a_1)$ gives $\mathscr{J}(b, a) \leq \mathscr{J}(b_1, a)$ since $b_1 = b + k$ for some k. Similarly $a_1 = a + m$ for some m, and $\mathscr{J}(b_1, a + m) \leq \mathscr{J}(b_1, a + m + 1)$ or $\mathscr{J}(b_1, a + m) = a + m + 1$ and $\mathscr{J}(b_1, a + m + 1) = a + m$ which gives $\mathscr{J}(b_1, a + m) \leq \mathscr{J}(b_1, a_1)$ since a_1 is a plus some natural number. □

The picture developed here is that an attractive interaction map is step-wise non-decreasing on all sets $S \times \{a\}$. It is step-wise non-decreasing on all sets $\{b\} \times S$ except for required decreases by exactly one on when we go directly from $\mathcal{U}$ to $\mathcal{D}$ from particle pairs (b, a) to $(b, a + 1)$, and on this jump, the value the map, $\mathscr{J}$, takes must go from $a + 1$ to a.

The interaction map of the multi-type contact process, Figure 2, is not attractive because $\mathscr{J}(2,0) = 2 > \min\limits_a\{a : (2,a) \in \mathcal{N} \cup \mathcal{D}\} = 1$. There are several ways to modify this interaction map into one which is attractive. First it seems that we would want to keep the births onto empty sites unchanged. But this means that $\mathscr{J}(2,1) = 2$ is necessary, which amounts to the invasive species seeing the native as non-different from and empty site. This then also forces the requirement that $\mathscr{J}(2,2) = 1$ or 2. It seems more natural to choose the latter, which would eliminate the crowding affect on the invader, although the first choice could also be reasonable, saying that when the invasive is too densely populated, it is easier for the native to come back in. Figure 3 gives these attractive versions of the multi-type contact process interaction map.

$_b\backslash^a$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	1

(a) Species 1 reinvades by a 2–2 interaction

$_b\backslash^a$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	2

(b) Species 2 immune to 2–2 interaction.

FIG 3. *Attractive interaction map modifications to the multi-type contact process*

15. Transition Rates. The transitions for these particle models have been split into two varieties. An *up* transition will be one where the interaction results in a larger particle value, a transition will be called *down* when the interaction results in a smaller particle value. The functions $r_u(\eta, x, y)$ and $r_d(\eta, x, y)$ will denote these transition rates respectively.

$$r_u(\eta, x, y) = \sum_{a \in S, b \in \mathcal{U}_a} \lambda_{ba}\phi(x,y) I_{\{(b,a)\}}(\eta(y), \eta(x)) \tag{15.1a}$$

$$r_d(\eta, x, y) = \sum_{a \in S, b \in \mathcal{D}_a} \lambda_{ba}\phi(x,y) I_{\{(b,a)\}}(\eta(y), \eta(x)) \tag{15.1b}$$

The summation is over the sets $\mathcal{U}_a = \{b \in S : \mathscr{J}(b,a) > a\}$ and $\mathcal{D}_a = \{b \in S : \mathscr{J}(b,a) < a\}$, which split the particles into those causing up and down transitions respectively. Null interactions are not accounted for here since a null transition does not change any state, thus their rates need no consideration.

We require that $\sum_y \phi(x,y) < \infty$ (or maybe need $\phi(x,y) = 0$ for $|x-y| > \rho$ for some $\rho < \infty$, or $\sum_y p(x,y) = 1$, where $p(x,y) = \phi(x,y)/\sum_y \phi(x,y)$) for any x, and typically $\max\phi = 1$, so that λ_{ba} will be the maximum transition rate for a particular interaction, and ϕ will be referred to as the *neighborhood mass function* since it defines how the strength of the interaction depends on a particular neighbor(or the probability that a particular neighbor is chosen to interact with suing $p(x,y)$).

The parameter λ_{ba}, is the rate parameter controlling the interaction that particle type $a \xrightarrow{b} \mathscr{J}(b,a)$ due to the presence of a particle of type b in its neighborhood. If $\lambda_{ba} \equiv 0$ for some a and b, then by convention, we put $(b,a) \in \mathcal{N}$, choosing $\mathscr{J}(b,a) = a$, for the process. This is so that all interactions will either have non-identically zero rates or be of the null variety.

The total transition rate will be denoted $r(\eta,x,y) = r_u(\eta,x,y) + r_d(\eta,x,y)$. This combining of up and down rates is appropriate since r_u and r_d can not be simultaneously nonzero for any given configuration due to the nature of the interaction map; an interaction is strictly either up, down, or null.

Example: For the nearest neighbor contact process, the transition rates are

$$\begin{aligned} r_u(\eta,x,y) &= \lambda_{10}\phi(x,y) I_{\{(1,0)\}}(\eta(y),\eta(x)) \\ r_d(\eta,x,y) &= \delta\phi(x,y) I_{\{(0,1)\}}(\eta(y),\eta(x)) + \delta\phi(x,y) I_{\{(1,1)\}}(\eta(y),\eta(x)) \\ &= \delta\phi(x,y) I_{\{1\}}(\eta(x)) \end{aligned}$$

where

$$\phi(x,y) = \begin{cases} 1 & \text{if} \quad |x-y| = 1 \\ 0 & \quad\text{otherwise} \end{cases} \tag{15.3}$$

There is a difference here that should be taken note of. The contact process is usually stated with a constant death rate of 1. In the current IMPS formulation, this amounts to $\delta = 1/4$ since we sum over the whole neighbor hood to get the total rate of death for particle type 1. If we let $\delta = 1$ in this formulation, then we are essentially rescaling time and would need to take note of how this changes the critical value for λ_{10}, which is usually just denoted by λ or β. Once again, this formulation may seem more complicated for a spin system, although consistent with, but later on it will become clear that it is appropriate for more complex systems.

The usual way to define the transitions for a process like this is to define f_b as the fraction of neighbors of particle type b for a finite neighborhood of size N. Assuming that $\phi(x,y) = 1/N$ on this neighborhood and is zero elsewhere gives the transition and rate shown in 15.4.

$$\begin{array}{cc} \text{Transition} & \text{Rate} \\ a \to c & \displaystyle\sum_{b:\mathscr{I}(b,a)=c} \lambda_{ba} f_b. \end{array} \tag{15.4}$$

This list should contain all transitions allowed by the model. The interaction map formulation is consistent with this due to the additivity of exponential rates.

Definition 16. *The generator for an IMPS is the closure in $C(X)$ of the operator defined on $D(X)$, by*

$$Gf(\eta) = \sum_{x,y} r(\eta,x,y)(f(\eta^{xy}) - f(\eta)) \tag{16.1}$$

The state η^{xy} represents the state η with the site x changed according to its interaction with the site y:

$$\eta^{xy}(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x \\ \mathscr{I}(\eta(y), \eta(x)) & \text{if} \quad z = x \end{cases}. \tag{16.2}$$

This notation is similar to what has been used before to denote a two particle jump or swap but differs in that it is only a change at a single site[3, 7]. The process is said to have *finite range* rates if the interactions occur over a finite neighborhood only, meaning that there exists a $\rho < \infty$ such that $\phi(x,y) = 0$ for $|x-y| > \rho$. It should be noted also that these rates are assumed to be bounded, $\sup_x \sum_y r(\eta,x,y) < \infty$. This holds true since

$$\sup_x \sum_y r(\eta,x,y) = \sup_x \sum_y \lambda_{\eta(y)\eta(x)}\phi(x,y) \le \max_{x,y} \phi(x,y) \cdot \max_{a,b} \lambda_{ba}.$$

This is obvious now since the λ_{ba}'s are only numbers and ϕ is bounded.

17. Coupling. Suppose we have two IMPS, $\eta_\mathbf{t}$ and $\xi_\mathbf{t}$, with transition rates $r_1(\eta,x,y)$ and $r_2(\xi,x,y)$ respectively, with $r_i(\cdot,x,y) = r_{iu}(\cdot,x,y) + r_{id}(\cdot,x,y)$. The coupling to follow is essentially the same as the Vasershtein coupling[11] given for spin systems, also known as the basic coupling[7]. The main desire is that the processes evolve together in such a way that keeps them closely related and each process adheres to the correct marginal rate. The coupled process, $(\eta_\mathbf{t}, \xi_\mathbf{t})$, will be a Feller process whose state space is $X \times X$.

The coupled process evolves according to the following rates:

$$(\eta,\xi) \to \begin{cases} (\eta^{xy},\xi^{xy}) & \text{at rate } \tilde{r}(\eta,\xi,x,y) \\ (\eta,\xi^{xy}) & \text{at rate } r_2(\xi,x,y)-\tilde{r}(\eta,\xi,x,y) \\ (\eta^{xy},\xi) & \text{at rate } r_1(\eta,x,y)-\tilde{r}(\eta,\xi,x,y) \end{cases}. \tag{17.1}$$

With $\tilde{r}(\eta,\xi,x,y) = \min(r_{1u}(\eta,x,y), r_{2u}(\xi,x,y)) + \min(r_{1d}(\eta,x,y), r_{2d}(\xi,x,y))$. So the generator for coupled processes is defined for $f \in C(X \times X)$, the space of continuous functions on $X \times X$, by

$$\tilde{G}f(\eta,\xi) = \sum_{x,y} (r_1(\eta,x,y)-\tilde{r}(\eta,\xi,x,y))(f(\eta^{xy},\xi)-f(\eta,\xi)) \tag{17.2a}$$

$$+\sum_{x,y} (r_2(\xi,x,y)-\tilde{r}(\eta,\xi,x,y))(f(\eta,\xi^{xy})-f(\eta,\xi)) \tag{17.2b}$$

$$+\sum_{x,y} \tilde{r}(\eta,\xi,x,y)(f(\eta^{xy},\xi^{xy})-f(\eta,\xi)) \tag{17.2c}$$

For this coupling, many results about the generator of the coupled process for spin systems hold as well. The following result needs a bit more work to prove though as there are many cases to check.

Theorem 17.1. *(Extension of Theorem 1.5 from Liggett IPS p127) Define the closed set* $K = \{(\eta,\xi) \in X \times X : \eta \le \xi\}$. *Suppose that the interaction map is attractive and thus satisfies*

(a) $\mathscr{J}$ *is non-decreasing on* $\mathcal{U}$
(b) $\mathscr{J}$ *is non-decreasing on* $\mathcal{D}$
(c) If $b_1 \le b_2$, $a_1 \le a_2$, $(b_1,a_1) \in \mathcal{U}$, *and* $(b_2,a_2) \in \mathcal{D} \cup \mathcal{N}$, *then* $\mathscr{J}(b_1,a_1) \le a_2$
(d) If $b_1 \le b_2$, $a_1 \le a_2$, $(b_1,a_1) \in \mathcal{U} \cup \mathcal{N}$, *and* $(b_2,a_2) \in \mathcal{D}$, *then* $a_1 \le \mathscr{J}(b_2,a_2)$

Furthermore suppose that whenever $\eta \le \xi$,

$$r_{1u}(\eta,x,y) \le r_{2u}(\xi,x,y) \quad \text{when} \quad \eta^{xy}(x) > \xi(x), \tag{17.3a}$$

$$r_{1d}(\eta,x,y) \ge r_{2d}(\xi,x,y) \quad \text{when} \quad \xi^{xy}(x) < \eta(x). \tag{17.3b}$$

Then for all $(\eta,\xi) \in K$ *and* $t \ge 0$,

$$P^{(\eta,\xi)}[(\eta_t,\xi_t) \in K] = 1. \tag{17.4}$$

Proof. Define $\mathscr{A} \subset C(X \times X)$ to be the subset of functions which are non-negative everywhere and 0 on K. Since $\mathscr{R}(I-\lambda\tilde{G}) = C(X \times X)$ (Prop. 2.8 p.15 Liggett IPS), there exists $h \in \mathscr{D}(\tilde{G})$ such that $(I-\lambda\tilde{G})h = f$ for each $f \in \mathscr{A}$ and $\lambda \ge 0$. Fix $f \in \mathscr{A}$, and define h by $h - \lambda\tilde{G}h = f$. Because K is compact, there exists an $(\eta,\xi) \in K$ where h achieves its maximum. The following cases will show that the generator applied to $h(\eta,\xi)$ is strictly non-positive.

Case 1. If $\xi(x) < \eta^{xy}(x)$, then $r_{1u}(\eta,x,y) \le r_{2u}(\xi,x,y)$ by (17.3a). Since

$$\tilde{r}(\eta,\xi,x,y) = \min(r_{1u}(\eta,x,y), r_{2u}(\xi,x,y)),$$

we get $r_1(\eta,x,y) - \tilde{r}(\eta,\xi,x,y) = 0$. So the problem transition, $(\eta,\xi) \to (\eta^{xy},\xi)$, occurs at rate zero by (17.1). The term (17.2a) disappears from the generator and all that is left are transitions which remain in the set K. Because (η,ξ) is where h achieves its maximum, the remaining terms of the generator, (17.2b) and (17.2c) are non-positive.

Case 2. If $\xi^{xy}(x) < \eta(x)$, $r_{1d}(\eta,x,y) \ge r_{2d}(\xi,x,y)$ by (17.3b). Then $r_2(\eta,x,y) - \tilde{r}(\eta,\xi,x,y) = 0$ since r_2 is the minimum here leading to $(\eta,\xi) \to (\eta,\xi^{xy})$ at rate zero. Once again, similar to the above calculation, the term (17.2b) is now zero. The remaining terms are non-positive since h achieves its maximum at (η,ξ).

Case 3. If $\xi(x) < \xi^{xy}(x)$ and $\eta(x) < \eta^{xy}(x)$, then $\eta^{xy}(x) \leq \xi^{xy}(x)$ by the first attractive interaction map property. Assuming that $\xi(x) < \eta^{xy}(x)$ would put us back in case 1 and we are done. If we assume that $\eta^{xy}(x) \leq \xi(x)$, then there are no problem transitions meaning that every possible coupled transition in (17.1) remains in the set K. Once again since (η, ξ) is where h attains its maximum, the generator remains non-positive in this case.

Case 4. If $\xi^{xy}(x) < \xi(x)$ and $\eta^{xy}(x) < \eta(x)$, then $\eta^{xy}(x) \leq \xi^{xy}(x)$ by the second attractive interaction map property. If $\xi^{xy}(x) < \eta(x)$, then we are back in case 2, otherwise all transitions keep us in K and the generator is non-positive.

Case 5. If $\eta(x) < \eta^{xy}(x)$ and $\xi^{xy}(x) \leq \xi(x)$, then $\eta^{xy}(x) \leq \xi(x)$ by the third attractive interaction map property. So $\tilde{r}(\eta, \xi, x, y) = 0$ since the minimum up transition rate is 0 as is the minimum down transition rate. This gives $(\eta, \xi) \to (\eta^{xy}, \xi^{xy})$ at rate zero. The remaining transitions keep us in K.

Case 6. If $\eta(x) \leq \eta^{xy}(x)$ and $\xi^{xy}(x) < \xi(x)$, then $\eta(x) \leq \xi^{xy}(x)$ by the fourth attractive interaction map property. So $\tilde{r}(\eta, \xi, x, y) = 0$ since the minimum up transition rate is 0 as is the minimum down transition rate. This gives and $(\eta, \xi) \to (\eta^{xy}, \xi^{xy})$ at rate zero once again remaining in K for the final case.

This shows that for all possible cases, $\lambda\tilde{G}h(\eta, \xi) \leq 0$ so that $h(\eta, \xi) \leq h(\eta, \xi) - \lambda\tilde{G}h(\eta, \xi) = f(\eta, \xi) = 0$. Since $\min_{(\zeta_1,\zeta_2)\in X\times X} h(\zeta_1, \zeta_2) \geq \min_{(\zeta_1,\zeta_2)\in X\times X} f(\zeta_1, \zeta_2)$ (Prop. 2.8 p.15 with Def. 2.1 p.12 Liggett IPS), we see that $h = 0$ on K, concluding that $h \in \mathscr{A}$. Since $(I - \lambda\tilde{G})^{-1}$ maps $\mathscr{A}$ to itself, so does $\tilde{S}(t)$ (Hille Yoshida Thm). Since this is true for any $f \in \mathscr{A}$, when $(\eta, \xi) \in K$, $P^{(\eta,\xi)}[(\eta_t, \xi_t) \in K] = 1$ for any t. □

With the assumptions of Theorem 17.1, if μ_1 and μ_2 are probability measures on X with $\mu_1 \leq \mu_2$, then $S_s(t)\mu_1 \leq S_2(t)\mu_2$ by Corollary 1.7 in Liggett IPS. This will be useful in the next section for proving the equivalence of monotonicity and attractiveness.

18. Monotonicity and Attractiveness.

Monotonicity for this type of process requires restrictions on the interaction map in addition to restrictions on the transitions rates.

DEFINITION 19. *An IMPS will be called attractive if it has an attractive particle interaction map and given $\eta \leq \xi$, the transition rates satisfy:*

(19.1a) $$r_u(\eta, x, y) \leq r_u(\xi, x, y) \quad \textit{when} \quad \mathscr{I}(\eta(y), \eta(x)) > \xi(x),$$

(19.1b) $$r_d(\eta, x, y) \geq r_d(\xi, x, y) \quad \textit{when} \quad \mathscr{I}(\xi(y), \xi(x)) < \eta(x).$$

Definition 19 is similar to the definition of attraction for spin systems given by Liggett[7]. The importance of the interaction map does not reveal itself for spin systems though, since the interactions are built into the flip rate. Separating the transitions rates from the interactions simplifies the system here.

To understand what the inequalities of Definition 19 mean, consider two ordered configurations, $\eta \leq \xi$. In the event that η could possibly jump above ξ at the site x, the latter must have a faster up transition rate(and be going to the same or a higher particle value). When the upper could possibly jump below the lower configuration, the latter must have a larger down transition rate. These definitions together will guarantee monotonicity for the process, and the next step is to show that they are indeed necessary.

THEOREM 19.1. *(Extension of Theorem 2.2 from Liggett IPS p.134): An IMPS is monotone if and only if it is attractive.*

Proof. Assuming the process is attractive and given two arbitrary and ordered initial distributions on X, $\mu_1 \leq \mu_2$, we will show that $\mu_1 S(t) \leq \mu_2 S(t)$ which is equivalent to monotonicity of the process by Definition **??**. For the coupled process (17.1), we are assured the existence of the initial distribution, ν on $X \times X$, according to Theorem **??**. Since $\tilde{S}(t)$ is the semigroup of the coupled process, we know $\nu\tilde{S}(t)(\{(\eta,\xi) : \eta \leq \xi\}) = 1$ by Theorem 17.1. The marginal distributions of $\nu\tilde{S}(t)$ are $\mu_1 S(t)$ and $\mu_2 S(t)$ which proves that $\mu_1 S(t) \leq \mu_2 S(t)$ by Theorem **??**, completing the proof that the attractive process is monotone.

Assuming that the process is monotone we must prove that it is attractive. Choose an x and b such that $\eta(x) \leq \xi(x) < b$. This will be used to prove conditions on up transitions. If no such b exists, there is no problem, as $\eta(x)$ cannot jump above $\xi(x)$. Similarly if there is a b such that $b \leq \eta(x) \leq \xi(x)$, then this fact can be used to show something about down transitions.

Start by defining the monotone function $f_{bx}(\eta) = I_{[b..n]}(\eta(x))$, the indicator on all particle values bigger than or equal to b, assuming that n is the maximum particle number. Applying the semigroup to this function, $S(t)f_{bx}$, again gives a monotone function for any $t \in [0,\infty)$ since the process is monotone. Note that in the cases we are considering, b is chosen so that $f_{bx}(\eta) = f_{bx}(\xi)$. This leads to the following calculation.

The generator and semigroup are related by:

$$Gf(\eta) = \sum_x \sum_y r(\eta,x,y)(f(\eta^{xy}) - f(\eta)) = \lim_{t\searrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}$$

For the monotone function $f_{b\tilde{x}}(\eta)$, we don't need to sum over x since it is fixed at $\tilde{x}$ giving:

$$Gf_{b\tilde{x}}(\eta) = \sum_y r(\eta,\tilde{x},y)(f_{b\tilde{x}}(\eta^{\tilde{x}y}) - f_{b\tilde{x}}(\eta)) = \lim_{t\searrow 0} \frac{S(t)f_{b\tilde{x}}(\eta) - f_{b\tilde{x}}(\eta)}{t}$$

Now that $\tilde{x}$ is fixed, we will drop the tildes. Because $f_{bx}(\eta) = f_{bx}(\xi)$, we get an inequality from the fact that $f_{bx}(\cdot)$ is monotone:

$$S(t)f_{b\tilde{x}}(\eta) \leq S(t)f_{b\tilde{x}}(\xi)$$

implies

$$\frac{S(t)f_{b\tilde{x}}(\eta) - f_{b\tilde{x}}(\eta)}{t} \leq \frac{S(t)f_{b\tilde{x}}(\xi) - f_{b\tilde{x}}(\xi)}{t}$$

which leads to

$$Gf_{bx}(\eta) \leq \lim_{t\searrow 0} \frac{S(t)f_{bx}(\xi) - f_{bx}(\xi)}{t} = \sum_{y\in N(x)} r(\xi,x,y)(f_{bx}(\xi^{xy}) - f_{bx}(\xi)) = Gf_{bx}(\xi)$$

So we get:

$$\sum_{y\in N(x)} r(\eta,x,y)(f_{bx}(\eta^{xy}) - f_{bx}(\eta)) \leq \sum_{y\in N(x)} r(\xi,x,y)(f_{bx}(\xi^{xy}) - f_{bx}(\xi)) \tag{19.2}$$

Which gives the total rate that η goes up into the set $[b,n]$ is less than or equal to the total rate that ξ goes up into the set $[b..n]$, and the total rate that $\eta(x)$ leaves the set $[b..n]$ (by going down) is greater than or equal to the total rate that $\xi(x)$ does the same.

Then, due to the type of rate function, $r(\eta,x,y) = \lambda_{\eta(y),\eta(x)}(x)\phi(x,y)$, (19.2) becomes

$$\sum_y \lambda_{\eta(y),\eta(x)}(x)\phi(x,y)(f_{bx}(\eta^{xy}) - f_{bx}(\eta)) \leq \sum_y \lambda_{\xi(y),\xi(x)}(x)\phi(x,y)(f_{bx}(\xi^{xy}) - f_{bx}(\xi)) \tag{19.3}$$

If we choose a particular neighbor of interest, say $\tilde{y}$ and make $\eta(y) = \eta(\tilde{y})$ and $\xi(y) = \xi(\tilde{y})$ for all $y \neq x$, (19.3) becomes

$$\lambda_{\eta(\tilde{y}),\eta(x)}(x)\sum_y \phi(x,y)(f_{bx}(\eta^{x\tilde{y}}) - f_{bx}(\eta)) \leq \lambda_{\xi(\tilde{y}),\xi(x)}(x)\sum_y \phi(x,y)(f_{bx}(\xi^{x\tilde{y}}) - f_{bx}(\xi)) \tag{19.4}$$

and then $\sum_y \phi(x,y)$ cancels out from both sides.

$$\lambda_{\eta(\tilde{y}),\eta(x)}(x)(f_{bx}(\eta^{x\tilde{y}}) - f_{bx}(\eta)) \leq \lambda_{\xi(\tilde{y}),\xi(x)}(x)(f_{bx}(\xi^{x\tilde{y}}) - f_{bx}(\xi))$$

then changing $\tilde{y}$ back into y since this did not depend on exactly which neighbor was singled out:

$$\lambda_{\eta(y),\eta(x)}(x)(f_{bx}(\eta^{xy}) - f_{bx}(\eta)) \leq \lambda_{\xi(y),\xi(x)}(x)(f_{bx}(\xi^{xy}) - f_{bx}(\xi)) \tag{19.5}$$

The following inferences come from looking at the possibilities for 19.5.

(i) If $\eta(x) \leq \xi(x) < b$, then
  (a) $\eta^{xy}(x) \geq b \Rightarrow \xi^{xy}(x) \geq b$
  (b) $\xi^{xy}(x) < b \Rightarrow \eta^{xy}(x) < b$
(ii) If $b \leq \eta(x) \leq \xi(x)$, then
  (a) $\xi^{xy}(x) < b \Rightarrow \eta^{xy}(x) < b$
  (b) $\eta^{xy}(x) \geq b \Rightarrow \xi^{xy}(x) \geq b$

Letting $b = \xi(x)+1$ in inference (i,a), we see that when $\eta^{xy}(x) > \xi(x)$, 19.5 becomes $\lambda_{\eta(y),\eta(x)}(x) \leq \lambda_{\xi(y),\xi(x)}(x)$ which is our first rate restriction. For the second rate restriction, let $b = \eta(x)$ in inference (ii,a), we see that when $\eta(x) > \xi^{xy}(x)$, 19.5 becomes $\lambda_{\eta(y),\eta(x)}(x) \geq \lambda_{\xi(y),\xi(x)}(x)$ finishing the rate restrictions for being an attractive process.

These inferences are true for any b, thus if both states are looking at possible up transitions, $\eta(x) < \eta^{xy}(x)$ and $\xi(x) < \xi^{xy}(x)$, then $\xi^{xy}(x) < \eta^{xy}(x)$ is not possible since we can let $b = \eta^{xy}(x)$ which violates both inferences under (i). Similarly if both states are looking at possible down transitions, $\eta^{xy}(x) < \eta(x)$ and $\xi^{xy}(x) < \xi(x)$, then $\xi^{xy}(x) < \eta^{xy}(x)$ is not possible since we can let $b = \xi^{xy}(x) + 1$ which violates both inferences under (ii). These give us the first two requirements for the interaction map to be attractive.

Assume that $\eta(x) < \eta^{xy}(x)$ and $\xi^{xy}(x) \leq \xi(x)$, and let $b = \xi(x)+1$. Then inference (i,b) is only satisfied if $\eta^{xy}(x) < b$. If $\eta(x) \leq \eta^{xy}(x)$ and $\xi^{xy}(x) < \xi(x)$, then letting $b = \eta(x)$. Inference (ii,b) is now only satisfied if $b \leq \xi^{xy}(x)$. Now the last two attractive interaction map requirements are met. □

The proof of Theorem 19.1 is somewhat more involved than the case for spin systems. This is because more complicated transitions are allowed which amount to "jump overs". Now this result applies to any interaction map particle system. The limitation should be noted that the neighborhood mass function must be the same for all species of particle for a given site. But spatial and temporal variations can be included and do not affect this result.

19.1. *Reordering the Particles.* If one develops a model which either does not have an attractive interaction map, or the rate restrictions which allow this model to be attractive are not desirable, a re-ordering of the particle values may give attractive model with more desirable rate restrictions. Biologically, the ordering of the particles may not necessarily have a meaning, it is purely a mathematical construction for the purpose of having a monotone model along with the benefit of using CFTP.

Interestingly enough, the multi-type contact process can be made attractive with a re-ordering. The main issue to resolve is that the interaction map is not attractive in this case, but making the permutation $\{0,1,2\} \rightarrow \{1,0,2\}$ in the sense that now $1 < 0 < 2$ our map is now attractive. To abate confusion on such a funny ordering of integers, we label particle 0 as one species of plant(species 0 to avoid confusion), particle 1 as an empty site, and particle 2 remains species 2. The rates are still the same: empty sites become species i at rate β_i and species i dies at constant rate δ_i. This information is summarized in Figure 4.

$b\backslash a$	0	1	2
0	δ_0	β_0	δ_2
1	δ_0	$\emptyset$	δ_2
2	δ_0	β_2	δ_2

$b\backslash a$	0	1	2
0	1	0	1
1	1	1	1
2	1	2	1

FIG 4. *Reordering of the multi-type contact process for attractiveness.*

This model now has an attractive interaction map and no rate restrictions. The intuition is usually to let 0 represent an empty site, but in this case it is beneficial to go against this standard by allowing us to label one species as "0" and the other as "2" with the empty site being represented by the particle value of 1. This gives us ordered extremal stationary distributions: $\nu_0 \le \delta_1 \le \nu_2$ where $\nu_i = \lim\limits_{t\to\infty} \delta_i S(t)$ is the invariant measure for species i in the supercritical case.

20. The Graphical Representation. Now we define a coupling of these processes constructed by way of Poisson point processes. This graphical representation was first introduced by Harris[4, 5] and subsequently developed by others[3, 7, 8]. This method is extremely useful since it allows multiple copies of the process to be constructed simultaneously.

For each site, x, and every y such that $\phi(x,y) > 0$, let U_{xy} and D_{xy} be two independent, identically distributed Poisson point processes on $(0,\infty)\times(0,\infty)$ with intensity equal to two-dimensional Lebesgue measure. Assume that our rates have an upper bound, c. For each x, define $\mathfrak{T}_x = \{T_{x,1} < T_{x,2} < ...\}$ by $T_{x,0} = 0$ and

$$T_{x,n} = \inf\{t > T_{x,n-1} : (v,t) \in \bigcup_y U_{xy} \cup D_{xy} \text{ for some } v \le c\}$$

which will represent the times at which transitions could possibly occur at the site x. This is a projection of a union of independent Poisson point processes, and the intensity measure of points in $\mathfrak{T}_x$ is $2c\#N(x,\rho)$ where $N(x,\rho) = \{y : |x-y| \le \rho \text{ and } \phi(x,y) > 0\}$, so $\mathfrak{T}_x$ is also a Poisson point process.

The graphical representation is created on a grid $\Lambda \times [0,\infty)$. At each point, t, in $\mathfrak{T}_x$, if $\exists u, y$ such that $(u,t) \in U_{xy}$ draw an arrow from y pointing to x with an open circle at x. If $\exists u, y$ such that $(u,t) \in D_{xy}$ draw an arrow from y pointing to x with a closed circle at x. Write the u values next to the tip of each arrow. Figure 5 give a possible realization of the graphical representation of this coupling for a one dimensional index set and rates bound above by $c = 3$.

These point processes are used to evolve the interaction map particle system. First assume we are given an initial state, η and that the rates are bounded above by c.

THEOREM 20.1. *The path η_t is constructed according to the following rules is the interaction map particle system with interaction map $\mathscr{J}$, and generator given by Definition 16.*

(a) Up transition rule: The particle at site, x, is replaced by the particle given by

$$\mathscr{J}(\eta_{t-}(y), \eta_{t-}(x))$$

at time $t \in \mathfrak{T}_x$ if there exists a u such that $(u,t) \in U_{xy}$, with

$$\mathscr{J}(\eta_{t-}(y), \eta_{t-}(x)) > \eta_{t-}(x),$$

and $u \le r_u(\eta_{t-}, x, y)$.

(b) Down transition rule: The particle at site, x, is replaced by the particle given by

$$\mathscr{J}(\eta_{t-}(y), \eta_{t-}(x))$$

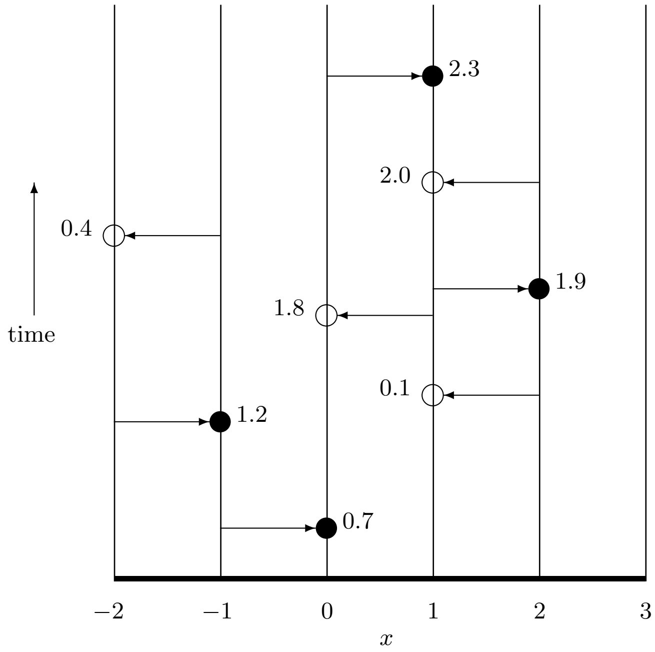

Fig 5. *A realization of the graphical representation of the point process coupling. All points with* $u \geq 3$ *have been left out. Closed and open circles represent possible down and up transition points respectively.*

at time $t \in \mathfrak{T}_x$*, if there exists a* u *such that* $(u, t) \in D_{xy}$*, with*

$$\mathscr{I}(\eta_{t-}(y), \eta_{t-}(x)) < \eta_{t-}(x),$$

and $u \leq r_d(\eta_{t-}, x, y)$.

The construction of these Poisson point processes and Theorem 20.1 is based upon Chapter 32 of Fristedt and Gray; see [3] for a rigorous proof. This is the basis for an accurate method of simulating these types of processes.

Theorem 20.2. *The graphical construction in Theorem 20.1 is monotone in the sense that it maintains the partial order of an attractive IMPS.*

Proof. This is proven by the fact that all up transitions preserve the order as do down transitions. The only time the partial ordering can be broken would be if up and down transitions which cross each other are allowed by the same point process, this is not the case here. ☐

So for any attractive interaction map particle system, it is easy to construct a monotone coupling. There are many possibilities that can be used as well. It is not always necessary to used U_{xy} and D_{xy}. Transitions can be grouped into point processes according to any criteria, they just can't be grouped together in such a way that allows an up transition for one configuration and a down transition for another if these two transitions break the ordering.

Theorem 20.3. *Consider two sets of parameters satisfying* $\lambda_{ba}^{(1)} \leq \lambda_{ba}^{(2)}$ *for all up transitions and* $\delta_{ba}^{(1)} \geq \delta_{ba}^{(2)}$ *for all down transitions. If two states satisfying* $\xi^{(1)} \leq \xi^{(2)}$ *are the initial states for the processes with the corresponding parameter sets above, then* $\xi_t^{(1)} \leq \xi_t^{(2)}$ *for all* $t \geq 0$ *for the above graphical construction.*

Proof. This is proved by Theorem 17.1. ☐

So an attractive interaction map process is monotone in each of its rate parameters.

21. Multiple Interaction Maps. It may be the case that more complicated interactions are desired such as allowing a certain transition to have a constant rate parameter in addition to depending on the neighborhood composition. As mentioned earlier, rare long range dispersal events may be desired to induce irreducibility. Take for example the grass–bushes–trees model described in Section 2.

In order to make this model attractive it needs two things. First we must reorder the particles as we did for the multi-type contact process, then we must introduce an extra point process for the births of trees. The basic interaction map, $\mathscr{J}_1$, and transition rates are given by Figure 6. This interaction map is the basic multi-type contact process map with births for species 2 removed and is attractive with no rate restrictions. Now we just need to account for this extra birth event. This amounts to including the extra interaction map, $\mathscr{J}_2$, given by Figure 7, and this map is attractive as well. There are only two actual transitions occurring in this map, the births of species 2. This is not a unique formulation, deaths could be distributed among both interaction maps and still maintain attractiveness. When simulating this process we need three Poisson Point Processes: $U =$ up transitions for basic multi-type contact process excluding species 2 births, $D =$ down transitions for multi-type contact process, and $B =$ birth events for species 2. Using this method allows the process to be attractive with no rate restrictions.

$_b\backslash^a$	0	1	2
0	δ_1	β_1	δ_2
1	δ_1	$\emptyset$	δ_2
2	δ_1	$\emptyset$	δ_2

$_b\backslash^a$	0	1	2
0	1	0	1
1	1	1	1
2	1	1	1

Fig 6. *Grass–bushes–trees initial rate parameters and interaction map,* $\mathscr{J}_1$.

$_b\backslash^a$	0	1	2
0	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	$\emptyset$
2	β_2	β_2	$\emptyset$

$_b\backslash^a$	0	1	2
0	0	1	2
1	0	1	2
2	2	2	2

Fig 7. *Extra rate parameters and interaction map,* $\mathscr{J}_2$*, for the birth of trees.*

It may seem cumbersome to think of the process this way. The idea is that you can have any arbitrary number of interaction maps. If each map is attractive, then a collection of rate restrictions allows the model to be monotone. Each map just needs to have distinct up and down point processes associated with it in the universal coupling.

Formulating a process with multiple interaction maps my actually relax some parameter restrictions for monotonicity. The equivalency of attractiveness and monotonicity will not be developed here, but a sufficiency condition is given.

Theorem 21.1. **Monotonicity Sufficiency Condition**: *Assume we have an interaction map particle system with multiple interaction maps,* $\mathscr{J}_1, \mathscr{J}_2, ..., \mathscr{J}_n$*, such that no two interaction maps have any non-null transitions in common. If each interaction map and the rate parameters associated with it are attractive, then* $\{U_1, D_1, U_2, D_2, ..., U_n, D_n\}$*, is a monotone coupling for the process. Where* U_i *and* D_i *are the point processes for the up and down transitions for interaction map* $\mathscr{J}_i$ *respectively.*

Proof. Each point process preserves the partial ordering of the state space, thus we have a monotone coupling, and therefore the process is monotone. □

This allows us to have interactions between two particle types give non-unique results. In other words, two particle types can interact with the resulting transition chosen from several possibilities. Only a sufficiency condition is given because assuming monotonicity and using the same methods as in Theorem 19.1 gives inequalities involving sums of parameters rather than individual parameters. This may allow relax the requirement that each individual map and its parameters be attractive. The grass–bushes–trees model in Figures 6 and 7 is attractive with no rate parameter restrictions according to this theorem.

The multi-type contact process with spontaneous births is formulated as usual, but with the inclusion of 2 extra point processes. One would represent spontaneous births for species 0, B_0, and the other for species 2, B_2. These extra point processes are formulated exactly as described above in the universal coupling. Two rates γ_0 and γ_2 represent the constant rates of spontaneous births for species 0 and 2 respectively. The same formulation of the graphical coupling applies here; if a point in B_i is encountered, then a 1 is allowed to become an i if $u \leq \gamma_i$. This process is still monotone with no parameter restrictions since the interaction maps of these spontaneous transitions are attractive and there are no "overjumps" to consider. The interactions maps and rates are given in Figures 8 and 9. It may be thought that this formulation is unnecessarily awkward, but it is consistent with the theory presented here.

$_b\backslash^a$	0	1	2
0	$\emptyset$	γ_0	$\emptyset$
1	$\emptyset$	γ_0	$\emptyset$
2	$\emptyset$	γ_0	$\emptyset$

$_b\backslash^a$	0	1	2
0	0	0	2
1	0	0	2
2	0	0	2

FIG 8. *Extra rates and interactions for spontaneous births of species 0.*

$_b\backslash^a$	0	1	2
0	$\emptyset$	γ_2	$\emptyset$
1	$\emptyset$	γ_2	$\emptyset$
2	$\emptyset$	γ_2	$\emptyset$

$_b\backslash^a$	0	1	2
0	0	2	2
1	0	2	2
2	0	2	2

FIG 9. *Extra rates and interactions for spontaneous births of species 2.*

With this interaction map formulation, monotone properties of multi-type contact processes can be determined quickly. This enables a quick assessment of whether or not monotone CFTP can be used as well.

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