Assignment 4
Law of Large Numbers & Central Limit Theorem

Math 363
November 18, 2012

1. Consider the function \( g(x) = 8 \tan(x/2)/((x + 1)e^{2x}) \)

(a) Estimate the integral \( \int_0^{\pi/2} g(x) \, dx \) using Monte Carlo simulations with 100 uniform random variables on the interval \([0, \pi/2]\).

(b) Repeat the estimate 100 times and use this to estimate the standard deviation of the error in the integral.

(c) If the standard deviation of the error is \( \sigma/\sqrt{n} \) for some \( \sigma \) use part (b) to estimate \( \sigma \).

(d) Combine the 100 estimates to give an estimate of the integral based on 10,000 uniform random variables.

(e) Use part (c) to estimate the standard deviation of the estimate.

(f) Use the \texttt{integrate} command to compute numerically the integral.

(g) Using your estimate for the standard deviation, discuss how close your estimate is to the numerical value.

2. Consider the function \( g(x) = \sqrt{1-x}/\cos(\pi x/2) \).

(a) Give a plot of \( g \) on the interval \([0, 1]\).

(b) Use simple Monte Carlo simulation to provide 100 estimates of \( \int_0^1 g(x) \, dx \) based on a sample of 200 uniform random variables. Find the mean and standard deviation of these simulations.

(c) Check that

\[
f_X(x) = \begin{cases} 
\frac{1}{2\sqrt{1-x}}, & 0 \leq x \leq 1, \\
0, & \text{otherwise}
\end{cases}
\]

is a valid density function.

(d) Find the distribution function and the probability transform.

(e) Use importance sampling to provide 100 estimates of \( \int_0^1 g(x) \, dx \) based on a sample of 200 random variables with proposal density \( f_X \). Find the mean and standard deviation of these simulations.

(f) Find the ratio of the standard deviations and the variances of these two estimates.

3. Let \( X \) be the value on an unfair die with mass functions.
\begin{align*}
\begin{array}{c|cccccc}
  x & 1 & 2 & 3 & 4 & 5 & 6 \\
  f_X(x) & 1/4 & 1/4 & 1/12 & 1/12 & 1/12 & 1/12
\end{array}
\end{align*}

(a) Find the mean and standard deviation for \( X \).

(b) Now assume that \( X_1, X_2, \ldots, X_{100} \) are independent with the mass function in part (a). Let \( \bar{X} = (X_1 + X_2 + \cdots + X_{100})/100 \). Find the mean and the standard deviation of \( \bar{X} \).

(c) Simulate \( \bar{X} \) 2000 times and compare the mean and standard deviation with the result in part (b).

(d) Estimate \( P\{\bar{X} > 3\} \) using the central limit theorem and compare this to the value given by the simulation.

(e) Estimate the value of \( x \) so that \( P\{\bar{X} < x\} = 0.25 \) using both the central limit theorem and the \texttt{quantile} command.

(f) Create a histogram for the simulations of \( \bar{X} \) and indicate the value in parts (d) and (e).

4. During the \texttt{log} phase of bacterial growth, the size of the colony grows exponentially. Let \( R \) be ratio of the biomass at time 1 hour to the initial biomass. Then,

\[ R = e^r \quad (1) \]

where \( r \) is the instantaneous growth rate. Assume that this ratio has mean \( \mu_R = 1.25 \) and standard deviation \( \sigma_R = 0.25 \).

(a) Perform this growth experiment 25 times and let \( \bar{R} \) be the average of these 25 ratios. What is the mean and standard deviation of \( \bar{R} \)?

(b) Use the relationship in (1) to give an estimate \( \hat{r} \) for the instantaneous growth rate.

(c) Use the delta method to estimate \( \sigma_{\hat{r}} \), the standard deviation of this growth rate.

5. The focal length \( f \) of an optical instrument is needed. This is determined by using the thin lens formula,

\[ \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f} \]

where \( r_1 \) is the distance from the lens to the object and \( r_2 \) is the distance from the lens to the real image of the object.

The distance \( r_1 \) is independently measured 9 times and \( r_2 \) is independently measured 64 times. The mean of the measurements is the actual distances, 15 centimeters and 10 centimeters, respectively. The standard deviation of the measurement is 0.2 centimeters for \( r_1 \) and 1 centimeter for \( r_2 \).

(a) Let \( \bar{R}_1 \) be the sample mean of the 9 measurements to the object. Estimate, using the central limit theorem, \( P\{\bar{R}_1 > 15.1\text{cm}\} \).

(b) Let \( \bar{R}_2 \) be the sample mean of the 64 measurements to the image. Estimate, using the central limit theorem, \( P\{\bar{R}_2 < 10.1\text{cm}\} \).

(c) How many measurements are needed so that \( P\{|\bar{R}_2 - 10\text{cm}| > 0.1\text{cm}\} \leq 0.02 \).

(d) For measurements \( r_{1,1}, \ldots, r_{1,9} \) and \( r_{2,1}, \ldots, r_{2,64} \), estimate the focal length using

\[ \frac{1}{\bar{r}_1} + \frac{1}{\bar{r}_2} = \frac{1}{\hat{f}}. \]

Use the delta method to give an estimate of the standard deviation of \( \hat{f} \).