1. Consider the function \( g(x) = \frac{10}{4 + 3x \sin(2x)} \)

(a) Estimate the integral \( \int_{0}^{\pi/2} g(x) \, dx \) using Monte Carlo simulations with 100 uniform random variables on \([0, \pi/2]\).

(b) Repeat the estimate 100 times and use this to estimate the standard deviation of the error in the integral.

(c) If the standard deviation of the error is \( \sigma/\sqrt{n} \) for some \( \sigma \) use part (b) to estimate \( \sigma \).

(d) Repeat the estimate 100 times using 1600 uniform random variables and check your estimate of the error with the estimate based on using 100 samples.

2. Consider the function \( g(x) = \sqrt{x} / \tan(x) \).

(a) Give a plot of \( g \) on the interval \([0, \pi/2]\).

(b) Use simple Monte Carlo simulation to provide 100 estimates of \( \int_{0}^{\pi/2} g(x) \, dx \) based on a sample of 200 uniform random variables. Find the mean and standard deviation of these simulations.

(c) Check that 
\[
    f_X(x) = \begin{cases} 
        \frac{1}{\sqrt{2\pi} x}, & 0 \leq x \leq \pi/2, \\
        0, & \text{otherwise}
    \end{cases}
\]

is a valid density function.

(d) Find the distribution function and the probability transform.

(e) Use importance sampling to provide 100 estimates of \( \int_{0}^{\pi/2} g(x) \, dx \) based on a sample of 200 random variables with proposal density \( f_X \). Find the mean and standard deviation of these simulations.

(f) Find the ratio of the standard deviations and the variances of these two estimates.

3. Let \( X_1, X_2, \ldots, X_{100} \) be independent discrete random variables with mass function for a weighted 8-sided die
\[
    f_X(x) = \frac{x}{36}, \quad x = 1, \ldots, 8
\]

(a) Find the mean and standard deviation for this mass function.

(b) Let \( \bar{X} = (X_1 + X_2 + \cdots + X_{80})/80 \). Find the mean and the standard deviation of \( \bar{X} \).
(c) Simulate $\bar{X}$ 1000 times and compare the mean and standard deviation with the result in part (b).

(d) Estimate $P\{5.8 < \bar{X} < 6.0\}$ using the central limit theorem and compare this to the value given by the simulation.

(e) Estimate the value of $x$ so that $P\{\bar{X} > x\} = 0.30$ using both the central limit theorem and the `quantile` command.

(f) Create a histogram for the simulations of $\bar{X}$ and indicate the value in parts (d) and (e).

4. The goal is to find the fraction of individuals that are recessive for a simple Mendelian trait. In this study, 320 individuals are chosen.

(a) If $p = 0.76$ is fraction of genes in the populations that carry the recessive trait, then give the mean and the standard deviation of the number of recessive alleles in the sample of 320 individuals. (Each individual has two alleles.)

(b) Let $X$ be the number of alleles possessing the recessive trait. Use this to estimate the probability that fewer than 475 alleles are recessive ($\{X \leq 475\}$) and the probability that more than 500 are recessive ($\{X > 500\}$). Use the continuity correction and compare it to the actual binomial probabilities.

(c) An individual expresses the recessive trait if both of its alleles are recessive. If the type of allele an individuals has is independent, what is the fraction $q$ as a function of $p$ that expresses the recessive trait?

(d) Let $\hat{q}$ be the observed fraction of individuals in the study that are recessive. Use the delta method to give the mean and standard deviation of a normal approximation to $\hat{q}$ based on the knowledge of $\hat{q}$.

5. The focal length $f$ of an optical instrument is needed. This is determined by using the thin lens formula,

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f},$$

where $r_1$ is the distance from the lens to the object and $r_2$ is the distance from the lens to the real image of the object.

The distance $r_1$ is independently measured 16 times and $r_2$ is independently measured 25 times. The mean of the measurements is the actual distances, 10 centimeters and 20 centimeters, respectively. The standard deviation of the measurement is 0.1 centimeters for $r_1$ and 0.6 centimeters for $r_2$.

(a) Let $\bar{R}_1$ be the sample mean of the 16 measurements to the object. Estimate, using the central limit theorem, $P\{R_1 > 10.1\text{cm}\}$.

(b) Let $\bar{R}_2$ be the sample mean of the 25 measurements to the image. Estimate, using the central limit theorem, $P\{R_2 > 20.1\text{cm}\}$.

(c) How many measurements are needed so that $P\{|\bar{R}_2 - 20\text{cm}| > 0.1\text{cm}\} \leq 0.02$.

(d) Estimate the focal length using

$$\frac{1}{\bar{R}_1} + \frac{1}{\bar{R}_2} = \frac{1}{F}.$$ 

Use the delta method to give an estimate of the standard deviation of $F$. 

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