1. The independent random variables $X_1, X_2, \ldots, X_n$ have common probability distribution function

$$F_X(x|\alpha, \beta) = \begin{cases} 
0 & \text{if } x < 0, \\
(x/\beta)^\alpha & \text{if } 0 \leq x < \beta, \\
1 & \text{if } x \geq \beta. 
\end{cases}$$

(a) Find the density of these random variables.

(b) The maximum likelihood estimator $\hat{\beta}$ for $\beta$ is the maximum value in the data. Find the maximum likelihood estimators for $\alpha$.

(c) The length (in millimeters) of cuckoos’ eggs found in hedge sparrow nests can be modeled with this distribution. Give the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ for the data

$$22.0 \quad 23.9 \quad 20.9 \quad 23.8 \quad 25.0 \quad 21.7 \quad 23.8 \quad 22.8 \quad 23.1 \quad 23.5 \quad 23.0$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of $Beta(\theta, 2)$ random variables. The density function

$$f_X(x|\theta) = c(\theta)x^{\theta-1}(1-x), \quad 0 \leq x \leq 1, \quad \theta > 0.$$ 

(a) Find the value of $c(\theta)$ so that $f_X(x|\theta)$ is a density function.

(b) Find the maximum likelihood estimator for $\theta$.

(c) Find the method of moments estimator of $\theta$.

(d) Use simulation to estimate the standard deviation of these estimators for $\theta = 1/5$ and $\theta = 5$.

(e) (Extra) Find the Fisher information and use the delta method to estimate the standard deviations of the two estimators.

3. The focal length of a lens in your lab has been mislabeled. You decide to make repeated measurements of the distance to the image and the object in order to estimate the focal length. Here $r_1$ is the distance from the lens to the object and $r_2$ is the distance from the lens to the real image of the object. Here are the measurements.

```r
> r1<-c(9.63, 9.925,10.67,10.30,9.73,9.60,10.49,9.43,9.69,10.20,9.74,9.44,9.33,10.60, + 10.01,9.78,10.39,10.08,10.04,9.55,10.52,10.24,9.47,9.86,9.68)
> r2<-c(9.97,9.90,9.95,9.94,10.07,10.14,10.07,10.16,9.88,10.06,10.17,10.13)
```

(a) Give a 98% confidence interval for $r_1$ and for $r_2$. 

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(b) The focal length \( f \) is determined by using the thin lens formula,
\[
\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f}.
\]
Give an estimate \( \hat{f} \) based on these measurements and the thin lens formula.

(c) Use the delta method to give the standard deviation of \( \hat{f} \).

(d) Use this to devise a 98\% confidence interval for \( f \).

4. In a study of human memory, Shachs (1967) demonstrated that people recall the meaning of verbal material, but tend to forget the exact word-for-word details. In this study, people read a passage of text. Then the people were shown a test sentence and asked whether or not the identical sentence has appeared in the text. In one condition, the test sentence was phrased differently, but had the same meaning as a sentence that was in the text.

(a) Write a hypothesis to test that the performance is significantly better than expected by chance.

(b) Out of 45 people, 27 correctly noticed that change in the sentence Test the hypothesis based on the binomial distribution and report a \( p \)-value.

(c) Is the test significant at the 10\%, 5\%, 1\% level?

(d) Does your conclusion change is 28 people notice the change? 29 people?

5. Recently, the Center for Disease Control and Prevention reported that 3820 children out of a sample of 377093 has been identified with an autism spectrum disorder

(a) Create a 98\% confidence interval for the proportion of children identified with an autism spectrum disorder

(b) To prepare for a future test on the diagnosis of autism, state a null and alternative hypothesis to test that the proportion of children identified with an autism spectrum disorder has increased over the value \( p_0 = \frac{3820}{337093} \).

(c) Give the value of the power function \( \pi(p) \) for \( p = 0.012, 0.013, 0.014, 0.015 \) with the choice of \( \alpha = 0.05 \) and a sample of size 10000.

(d) Compute the power function for these values if we increase the sample to 25000.

(e) Give a sketch of the two power curves and explain why these values increased with increased sample size.

6. The body temperature in degrees Fahrenheit of 52 randomly chosen healthy adults is measured with the following summary of the data:
\[
n = 52, \quad \bar{x} = 98.2846.
\]
Assume a standard deviation of \( \sigma = 0.68 \).

(a) Are the necessary conditions for constructing a valid \( z \)-interval satisfied? Explain.

(b) Find a 98\% confidence interval for the mean body temperature and explain its meaning.

(c) Give a hypothesis test for a mean body temperature of 98.6\(^\circ\) Fahrenheit versus the alternative of 98.2\(^\circ\) Fahrenheit. Use the information above to evaluate a test with significance level \( \alpha = 0.02 \).

(d) Find the power of the test at the parameter value \( \mu = 98.2 \) and indicate this value using the cutoff value for the test and drawing the sample distribution for the null and alternative hypothesis.