Overview of Calculus

June 6, 2016

1 Limits

Calculus begins with the notion of **limit**. In symbols,

$$\lim_{x \to c} f(x) = L$$

In words, however close you demand that the function f evaluated at x, $f(x)$, to be to the limit L (usually called ϵ), we can determine now close x must be to c (usually called δ) to meet the demand.

A function f is called **continuous** at c if

$$\lim_{x \to c} f(x) = f(c).$$

f is called **continuous** if it is continuous for all x in its domain.

2 Slopes and Deriviatives

Lines have **slopes**. If

$$f(x) = mx + b,$$

then the slope

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{(m(x + \Delta x) + b) - (mx + b)}{\Delta x} = m.$$

Other than lines the **difference quotient**

$$\frac{f(x + \Delta x) - f(x)}{\Delta x},$$

called the slope of a **secant line**, is not constant. If this quotient has a limit at $\Delta x \to 0$,

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

then we say that f is **differentiable** at x. We write the limit

$$f'(x), \quad \frac{df}{dx}(x), \quad Df(x), \quad \text{or} \quad \frac{dy}{dx}.$$

and say the **derivative** of f at x. f is called **differentiable** if it is differentiable for all x in its domain. for example, for x near a value x_0, we have the linear approximatin

$$f(x) \approx f(x_o) f'(x_0)(x - x_0).$$

If $f'(x)$ is positive (negative), the we say that f is increasing (decresing) at x.

We use several phrases to provide an intuitive meaning to the derivative. For example, the derivative $f'(x)$ is

- the **instantaneous rate of change** of f at x.
- the slope of the **tangent line** of f at x.

For example, if $x(t)$ is the position of an object at time t, then $v(t) = x'(t)$ is the velocity and $a(t) = v'(t)$ is the acceleration at time t.

3 Rules for Differentiation

Let f and g be differentiable functions and b and b be constants

- The **linearity of the derivative** (Sum rule)

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x)$$

- **Product rule**

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$$

- **Quotient rule**

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

- **Chain rule**

$$(f \circ g)'(x) = \frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$$

4 Derivatives of Common Functions

- **Powers** For any real number p,

$$\frac{d}{dx}x^p = px^{p-1}.$$

- **Natural logarithm**

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

- **Exponential functions** For any real number $b > 0$

$$\frac{d}{dx} b^x = b^x \ln b,$$

In particular if $b = e$, Euler's constant,

$$\frac{d}{dx} e^x = e^x,$$

- **Trigonometric functions**

$$\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x, \quad \frac{d}{dx} \tan x = \sec x = \frac{1}{\cos^2 x}.$$

5 Definite Integrals

We will let f be a continuous and bounded function on the interval $[a, b]$, the goal is to define the integral

$$\int_a^b f(x)dx$$

in such a way that its intuitive meaning for positive functions is that the integral is the area below the function f, above the x-axis, to the fight of the vertical line $x = a$ and to the right of the line $x = b$. (The function that appears in the integral is called the **integrand**.)

To define these sums, first divide the interval into n contiguous subintervals of length

$$\Delta x = \frac{b-a}{n}.$$

For x_i^* in the i-th subinterval, $i = 1, 2, \ldots, n$, a Riemann sum

$$\sum_{i=1}^n f(x_i^*)\Delta x.$$

We focus on two such types of sums, namely, upper and lower **Riemann sums**. To define these, choose the minimum value m_i and the maximum value M_i for the i-th subinterval, $i = 1, 2, \ldots, n$. Then the lower Riemann sum

$$\sum_{i=1}^n f(m_i)\Delta x.$$

is the area of n rectangles that sit on the y-axis underneath the graph of f. The upper Riemann sum

$$\sum_{i=1}^n f(m_i)\Delta x.$$

is the area of n rectangles that sit on the y-axis above the graph of f.

Thus, the lower Riemann sum is an underestimate of the integral and the upper Riemann sum is an overestimate. However, as we refine the estimates, the lower Riemann sum increases and the upper Riemann decreases. Moreover, the difference between these two sums converges to 0 in the limit as $\Delta x \to 0$.

6 Fundamental Theorem of Calculus

We need a practical way to compute the integral that is more efficient than computing Riemann sums. We begin by defining a function

$$A(x) = \int_a^x f(t)dt,$$

the integral up to a value x. Next, let's examine the chenge in F over a small interval of length Δx.

$$A(x + \Delta x) - A(x) = \int_x^{x+\Delta x} f(t)dt \approx f(x)\Delta x,$$

and

$$\frac{A(x + \Delta x) - A(x)}{\Delta x} \approx f(x).$$

This approximation becomes an equality in the limit as $\Delta x \to 0$. Thus,

$$A'(x) = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = f(x).$$

We call A an **antiderivative** of f. If F is another antideriviative of f, then

$$F'(x) - A'(x) = f(x) - f(x) = 0.$$

The only function that are zero are the constant functions. In this case, the constant

$$c = F(a) - A(a) = F(a).$$

and

$$F(b) - F(a) = A(b) = \int_a^b f(x)dx.$$

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Written in this way, we call this a **definite integral**. Antiderivatives are also called **indefinite integrals** and are written

$$\int f(x)dx = F(x) + c$$

The addition of the constant c is written to remind us that antiderivatives are a family of functions, each one differs from another by a constant function.

7 Techniques for Integration

Not surprisingly, rules for taking derivatives provides techniques for integration, Again, let f, g, u, v, and w be continuous functions and b and b be constants

- The **linearity of the integral**

$$\int (af(x) + bg(x)) = a \int f(x)dx + b \int g(x)dx + c$$

- **Integration by parts**

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx + c$$

- w **substitution**

$$\int f(w(x))w'(x)dx = \int f(w)dw + c$$

8 Examples

- **Integration by parts**

1.
$$\int xe^{-x}dx = x(-e^{-x}) - \int(-e^{-x})(1)dx = -xe^{-x} - e^{-x} + c = -(x+1)e^{-x} + c$$

$$\begin{array}{ll} u(x) = x & v(x) = -e^{-x} \\ u'(x) = 1 & v'(x) = e^{-x} \end{array}$$

2.
$$\int \ln x dx = x\ln x - \int x \cdot \frac{1}{x}dx = x\ln x - \int 1dx = x\ln x - x = x(\ln x - 1) + c$$

$$\begin{array}{ll} u(x) = \ln x & v(x) = x \\ u'(x) = \frac{1}{x} & v'(x) = 1 \end{array}$$

- w **substitution**

1.
$$\int 2\cos x \sin x dw = \int 2wdw = w^2 + c = \sin^2 x + c$$

$$w(x) = \sin x \quad w'(x) = \cos x$$

or

2.
$$\int 2\cos x \sin x dw = -\int 2wdw = -w^2 + c = -\cos^2 x + c$$

$$w(x) = \cos x \quad w'(x) = -\sin x$$

Notice that the two antiderivatives

$$\sin^2 x - (-cos^2 x)$$

differ by a constant, namely $c = 1$.