Basics Principles of Counting

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Suppose that two experiments are to be performed.

- Experiment 1 can have n_1 possible outdomces and
- for each outcome of experiment 1, experiment 2 has n_2 possible outcomes.

Then together there are $n_1 n_2$ possible outcomes.

Exercise 1. *Generalize this basic principle of counting to k experiments.*

1 Permutations

Assume that we have a collection of n objects and we wish to make an **ordered arrangement** of k of these objects. Using the generalized principle of counting, the number of possible outcomes is

$$n \times (n-1) \times \cdots \times (n-k+1).$$

We will write this as $(n)_k$ and say n **falling** k.

Example 2 (birthday problem)**.** *In a list the birthday of k people, there are 365^k possible lists (ignoring leap year births) and*

$$(365)_k$$

possible lists with no date written twice. Thus, the probability, under equally likely outcomes, that no two people on the list have the same birthday is

$$\frac{(365)_k}{365^k}$$

and, under equally likely outcomes,

$$P\{\textit{at least one pair of individuals share a birthday} = 1 - \frac{(365)_k}{365^k}$$

For example

k	5	10	15	20	22	23	25	30	40	50	100
probability	0.027	0.117	0.253	0.411	0.476	0.507	0.569	0.706	0.891	0.970	0,994

The ordered arrangement of all n objects is

$$(n)_n = n \times (n-1) \times \cdots \times 1 = n!,$$

n **factorial**. We take $0! = 1$.

Exercise 3.

$$(n)_k = \frac{n!}{(n-k)!}.$$

2 Combinations

Write

$$\binom{n}{k}$$

for the number of number of different groups of k objects that can be chosen from a collection of n.

Theorem 4.

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Here is an example of a combinatorial proof.

We will form an ordered arrangement of k objects from a collection of n by:

1. First choosing a group of k objects.
   The number of possible outcomes for this experiment is $\binom{n}{k}$.
2. Then, Arranging this k objects in order.
   The number of possible outcomes for this experiment is $k!$.

So, by the basic principle of counting,

$$(n)_k = \binom{n}{k} \times k!.$$

Now complete the proof by dividing both sides by $k!$.

Example 5. *In 100 tosses of a coin, there are*

$$\binom{100}{67}$$

outcomes that have 67 heads. Thus, the probability of 67 heads in 100 coin tosses

$$\frac{\binom{100}{67}}{2^{100}}$$

Exercise 6 (Capture-Recapture)**.** *In a population of unknown size n, t individuals are* **tagged** *and returned. Later c are captured in such a way that all*

$$\binom{n}{c}$$

possible groups are equally likely. Among this group r have tags and hence are **recaptured**. *The number of ways to collect these data are*

$$\binom{t}{r} \times \binom{n-t}{c-r}.$$

So, under equally likely outcomes,

$$p_n = P_n\{r \text{ recaptured}\} = \frac{\binom{\text{t}}{\text{r}} \times \binom{\text{n}-\text{t}}{\text{c}-\text{r}}}{\binom{\text{n}}{\text{c}}}.$$

Find n that maximizes this probability. (Hint: For what values of n is $p_n/p_{n-1} > 1$.)

Example 7. *A standard* **poker hand** *consists of 5 cards from a deck of 52. Thus, there are*

$$\binom{52}{5}$$

poker hands. A **full house** *consists of a pair and three of a kind, Thus, there are*

$$13\binom{4}{2}12\binom{4}{3}$$

full houses. Consequently, the probability of a full house is

$$\frac{13\binom{4}{2}12\binom{4}{3}}{\binom{52}{5}}.$$

Exercise 8 (binomial theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Theorem 9 (Pascal's triangle)**.**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

To establish this identity, distinguish one of the n objects in the collection.

1. If the distinguished object is the group, then we must choose $k-1$ from the remaining $n-1$ objects. Thus, $\binom{n-1}{k-1}$ groups have the distinguished object.

2. If the distinguished object is not the group, then we must choose k from the remaining $n-1$ objects. Thus, $\binom{n-1}{k}$ groups do not have the distinguished object.

3. These choices of groups of no overlap,

3 Multinomial coefficients

If we want to divide n objects into r groups of size $n_1, n_2, \dots, n_r$, then

$$n_1 + n_2 + \cdots + n_r = n.$$

To determine the number of different choice, note that by the generalization of the basic principle of counting:

- We have $\binom{n}{n_1}$ possible choices for the first group.
- For each choice for the first group, we have $\binom{n-n_1}{n_2}$ possible choices for the second group.
- For each choice for the second group, we have $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group.

Thus the total number of choices is

$$\begin{aligned} & \binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} \times \cdots \times \binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\ = \quad & \frac{n!}{n_1!(n-n_1)!} \times \frac{n-n_1}{n_2!(n-n_1-n_2)!} \times \frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{n_r!0!} \\ = \quad & \frac{n!}{n_1!n_2!\cdots n_r!} \end{aligned}$$

We shall denote this

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

Example 10. *In 12 rolls of the dice, there are 6^{12} different outcomes. The number of outcomes in which each number appears twice is*

$$\binom{12}{2,2,2,2,2,2} = \frac{12!}{(2!)^6}.$$

Thus, the probability of this event is

$$\frac{12!}{2^6 6^{12}} = 0.0034.$$